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THEORY OF MAXIMA AND MINIMA

BY

HARRIS HANCOCK, PH.D. (BERLIN), DR.SC. (PARIS)

PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF CINCINNATI

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PREFACE

Mathematicians have always been occupied with questions of maxima and minima. With Euclid one of the simplest problems of this character was: *Find the shortest line which may be drawn from a point to a line,* and in the fifth book of the conics of Apollonius of Perga occur such problems as the *determination of the shortest line which may be drawn from a point to a given conic section.*

It is thus seen that a sort of theory of maxima and minima was known long before the discovery of the differential calculus, and it may be shown that the attempts to develop this theory exercised considerable influence upon the discovery of the calculus. Fermat, for example, after making numerous restorations of two books of Apollonius, often cites this old geometer in his "*method for determining maximum and minimum,*" *1638,* a work which in some instances is so closely related to the calculus that Lagrange, Laplace, Fourier, and others wished to consider Fermat as the discoverer of the calculus. This he probably would have been had he started from a somewhat more general point of view, as in fact was done by Newton (*Opuscula Newtoni, I,* 86–88).

Maclaurin (*A Treatise of Fluxions,* Vol. I, p. 214. 1742), wrote: "There are hardly any speculations in geometry more useful or more entertaining than those which relate to maxima and minima. Amongst the various improvements that began to appear in the higher parts of geometry about a hundred years ago, Mr. de Fermat proposed a method for finding the maxima and minima. How the methods that were then invented for the mensuration of figures and drawing tangents to curves are comprehended and improved by the method of Fluxions, may be understood from what has already been demonstrated. A general way of

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resolving questions concerning maxima and minima is also derived from it, that is so easy and expeditious in the most common cases, and is so successful when the question is of a higher degree, when the difficulty is greater and other methods fail us, that this is justly esteemed one of the most admirable applications of Fluxions."

The theory of maxima and minima was rapidly developed along the lines of the calculus after the discovery of the latter. Mathematicians were at first satisfied with finding the necessary conditions for the solution of the problem. These conditions, however, are seldom at the same time sufficient. In order to decide this last point, the discovery of further algebraic means was necessary. Descartes had already remarked, in a letter of March 1, 1638, that Fermat's rule for finding maxima and minima was imperfect; and we shall see that many imperfections still existed for a long time after the invention of the calculus by Newton.

As introductory to a course of lectures on the calculus of variations, I have for a number of years given a brief outline of the theory of maxima and minima. This outline is founded on the lectures that were presented by the late Professor Weierstrass in the University of Berlin. It treats the ordinary cases; that is, where the functions are everywhere regular and where the forms are either definite or indefinite. It was published as a bulletin of the University of Cincinnati in 1903. At that time I expected to publish another bulletin which was to treat the more special cases; for example, where only one-sided differentiation enters, the "ambiguous case," where the form is semi-definite, etc. A treatment of these cases, the extraordinary cases, required more study than was anticipated. The bulletin has consequently been delayed so long that I have concluded to give an entirely new exposition of the whole theory.

In the preface to the German translation by Bohlmann and Schepp of Peano's *Calcolo differenziale e principii di calcolo integrale*, Professor A. Mayer writes that this book of Peano not only is a model of precise presentation and rigorous deduction, whose propitious influence has been unmistakably felt upon

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almost every calculus that has appeared (in Germany) since that time (1884), but by calling attention to old and deeply rooted errors it has given an impulse to new and fruitful development.

The important objection contained in this book (Nos. 133–136) showed unquestionably that the entire former theory of maxima and minima needed a thorough renovation; and in the main Peano's book is the original source of the beautiful and to a great degree fundamental works of Scheeffer, Stolz, Victor v. Dantscher, and others, who have developed new and strenuous theories for *extreme* values of functions. Speaking for the Germans, Professor A. Mayer, in the introduction to the above-mentioned book, declares that there has been a long-felt need of a work which, for the first time, not only is free from mistakes and inaccuracies that have been so long in vogue but which, besides, so incisively penetrates an important field that hitherto has been considered quite elementary.

To a considerable degree these inaccuracies are due to one of the greatest of all mathematicians, Lagrange, and they have been diffused in the French school by Bertrand, Serret, and others. We further find that these mistakes are ever being repeated by English and American authors in the numerous new works which are constantly appearing on the calculus.

It seems, therefore, very desirable in the present state of mathematical science in this country that more attention be given to the theory of maxima and minima; for it has a high interest as a topic of pure analysis and finds immediate application to almost every branch of mathematics.

I have therefore prepared the present book for students who wish to take a more extended course in the calculus as introductory to graduate work in mathematics. I do not believe in making university students study abstruse theories in foreign languages, and in this treatise it will be found that the pedagogical side of the presentation is insisted upon; for example, the Taylor development in series is given under at least half a dozen different forms.

HARRIS HANCOCK

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CONTENTS

CHAPTER I

FUNCTIONS OF ONE VARIABLE

I. Ordinary Maxima and Minima

SECTION PAGE

1. *Greatest* and *smallest* value of a function in a fixed interval; *maximum* and *minimum*. Several maxima or minima in the same interval. *Ordinary* and *extraordinary*, *proper* and *improper* maxima and minima. The *extremes* of functions; *upper* and *lower* limits; *absolute* and *relative* maxima and minima 1
2. Criterion for maxima or minima values 2
3. Other theorems 4
4. Résumé of the above expressed in a somewhat different form 5

II. Extraordinary Maxima and Minima

A. Functions which have Derivatives only on Definite Positions

5. *Left-hand* and *right-hand* differential quotients 6
6. Criteria as to whether a root of the equation $f'(x) = 0$ offers an extreme of the function $f(x)$. Statement and proofs of certain theorems . . 7

B. Functions which have only One-Sided Differential Quotients of a Certain Order for a Value $x = x_0$

7. Theorems and their proofs 11

C. Upper and Lower Limits of a One-Valued Function which is Continuous for Values of the Argument within a Definite Interval

8. *Greatest* and *least* values; *upper* and *lower* limits. Examples 12
9. An interesting example given by Liouville 13
10. Examples in which the functions are discontinuous; cases in which the upper and lower limits are *not* reached 14

Problems 15

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CHAPTER II

FUNCTIONS OF SEVERAL VARIABLES

I. Ordinary Maxima and Minima

Preliminary Remarks

SECTION PAGE

11. *Proper* and *improper* extremes 17
12. Criteria for maxima and minima 17
13. *Definite*, *semi-definite*, and *indefinite forms* 19
14. Criterion for a definite quadratic form 19

II. Relative Maxima and Minima

15. Statement of the problem. Derivation of the conditions 21
Problems 22

CHAPTER III

FUNCTIONS OF TWO VARIABLES

I. Ordinary Extremes

16. Definitions and derivation of the required conditions. Statement of different cases after Goursat 23
17. The indefinite case 25
18. The definite case 25
19. Summary of the results derived for the two cases above. Example taken from the theory of least squares 26
Problems 27

Introduction to the Ambiguous Case $B^2 - AC = 0$

20–23. Exposition of the attending difficulties which are illustrated by means of geometric considerations due to Goursat 27
24. The classic example of Peano showing when we may and may not expect extremes 31

II. Incorrectness of Deductions made by Earlier and Many Modern Writers

25. The Lagrange fallacy followed by Bertrand, Serret, Todhunter, etc. Geometric explanation of this fallacy 33

III. Different Attempts to improve the Theory

26. Failure in deriving the required conditions by studying the behavior of the function upon algebraic curves 35
27. Statement of the Scheeffer method 37
28. The method of Von Dantscher outlined 39
29. The Stolzian theorems 39

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CHAPTER IV

THE SCHEEFFER THEORY

I. General Criteria for a Greatest and a Least Value of a Function of Two Variables; in particular the Extraordinary Extremes

SECTION PAGE

30. Stolz's analytic proof of the Scheeffer theorem 43
31. Stolz's added theorem by which the nonexistence of extremes may be often ascertained 45
32. Scheeffer's geometric proof of his own theorem 46
33. Failure of this theorem 48

II. Homogeneous Functions

34. Scheeffer's criteria of extremes for such functions 49

III. Extremes for Functions that are not Homogeneous

35. In particular the case where the terms of lowest dimension constitute a semi-definite form. *Extreme curves* 52
36. The criteria established 54
37–40. A direct and more practical method due to Stolz of deriving the above results 55
41. Exceptional cases 58
Examples 59

IV. The Method of Victor v. Dantscher

42. The general method outlined and applied to the definite and indefinite cases 62
43–44. Treatment of the semi-definite case 64
Problems 69

V. Functions of Three Variables

Treatment in Particular of the Semi-definite Case

45. Extensions of the theorems and proofs given by Stolz and Scheeffer for functions of two variables 70
46. Evident generalizations of §§ 37–41 71
Problems 72

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CHAPTER V

MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES THAT ARE SUBJECTED TO NO SUBSIDIARY CONDITIONS

I. Ordinary Extremes

SECTION PAGE

47. Nature of the functions under consideration. Definition of *regular functions* 73
48. Definition of *maxima* and *minima* of functions of one and of several variables 74
49. The problem of this chapter proposed. Taylor's theorem for functions of one variable 75
50. Taylor's theorem for functions of several variables 77
51. The usual form of the same theorem 79
52. A condition of maxima and minima of such functions 81

II. Theory of the Homogeneous Quadratic Forms

53. No maximum or minimum value of the function can enter when the corresponding quadratic form is *indefinite*. When is a quadratic form a *definite* form? 82
54. Some properties of quadratic forms. The condition that the quadratic form $\phi(x_1, x_2, \cdots, x_n) = \Sigma A_{\lambda\mu} x_\lambda x_\mu$ be expressible as a function of $n - 1$ variables 83
55-61. Every homogeneous function of the second degree $\phi(x_1, x_2, \cdots, x_n)$ may be expressed as an aggregate of squares of linear functions of the variables 85
62. The question of § 53 answered 91

III. Application of the Theory of Quadratic Forms to the Problem of Maxima and Minima stated in §§ 47–51

63. Discussion of the restriction that the definite quadratic form must only vanish when all the variables vanish. The problem of this chapter completely solved 92
64. Upper and lower limits 94

CHAPTER VI

THEORY OF MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES THAT ARE SUBJECTED TO SUBSIDIARY CONDITIONS. RELATIVE MAXIMA AND MINIMA

65. The problem stated 96
66. The natural way to solve it 96
67-70. Derivation of a fundamental condition 96
71. Another method of finding the same condition 98

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SECTION PAGE

72. Discussion of the restrictions that have been made 100
73. A geometrical illustration of these restrictions 101
74. Establishment of certain criteria 102
75. Simplifications that may be made 102
76. More symmetric conditions required 103

I. Theory of Homogeneous Quadratic Forms

77. Addition of a subsidiary condition 103
78, 79. Derivation of the fundamental determinant Δe and the discussion of the roots of the equation $\Delta e = 0$, known as the equation of secular variations 104
79. The roots of this equation are all real 107
80–82. Weierstrass's proof of the above theorem 108
83, 84. An important lemma 109
85. A general proof of a general theorem in determinants 111
86. The theorem proved when the variables are subjected to subsidiary conditions 112
87. Conditions that the quadratic form be invariably positive or invariably negative 114

II. Application of the Criteria just found to the Problem of this Chapter

88. The problem as stated in § 65 solved and formulated in a symmetric manner 114
89. The results summarized and the general criteria stated 115
90. Discussion of the geometrical problem: *Determine the greatest and the smallest curvature at a regular point of a surface.* Derivation of the characteristic differential equation of *minimal surfaces* . 116
91. Solution of the geometrical problem: *From a given point to a given surface draw a straight line whose length is a minimum or a maximum* 123
92. Brand's problems. Two problems taken from the theory of light. Reflection. Refraction 126

CHAPTER VII

SPECIAL CASES

I. The Practical Application of the Criteria that have been hitherto given and a Method founded upon the Theory of Functions, which often renders unnecessary these Criteria

93. Difficulties that are often experienced. Fallacies by which maxima and minima are established when no such maxima or minima exist 134

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SECTION PAGE

94. Definitions: *Realm.* A *position.* An *n-ple multiplicity. Structures.* A position defined which lies within the *interior* of a definite realm, on the *boundary*, or *without* this realm . . . 135
95. Statement of two important theorems in the theory of functions 136
96. *Upper* and *lower* limits for the values of a function. *Asymptotic approach.* Geometrical and graphical illustrations 137
97. Digression into the theory of functions 138

II. Examples of Improper Extremes

98. Cases where there are an infinite number of positions on which a function may have a maximum value 140
99. Reduction of such cases to the theory of maxima and minima proper. The derivatives of the first order must vanish . . . 141
100. The results that occur here are just the condition which made the former criteria impossible 142
101. The previous investigations illustrated by the problem: *Among all polygons which have a given number of sides and a given perimeter, find the one which contains the greatest surface-area* . 142
102–107. Solution of the above problem 143
102. Cremona's criterion as to whether a polygon has been described in the positive or negative direction 143
Problem 147
108. A problem due to Hadamard 147

III. Cases in which the Subsidiary Conditions are not to be regarded as Equations but as Limitations

109. Examples illustrating the nature of the problem when the variables of a given function cannot exceed definite limits . . . 149
110. Reduction of two inequalities to one 149
111. Inequalities expressed in the form of equations 150
112. Examples taken from mechanics 150

Gauss's Principle

113. Statement of this principle 151
114. Its analytical formulation 152
115. By means of this principle all problems of mechanics may be reduced to problems of maxima and minima 152
116. Proof of a theorem in the theory of functions 153

The Reversion of Series

117–126. Proof and discussion of this theorem. Determination of upper and lower limits for the quantities that occur 154
123. Unique determination of the system of values of the x's that satisfy the equations 159

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SECTION PAGE

124. If the y's become infinitely small with the x's, the x's become infinitely small with the y's. The x's are continuous functions of the y's * 160
125. The x's, considered as functions of the y's, have derivatives which are continuous functions of the y's. The existence of the first derivatives 162
126. The x's, expressed in terms of the y's, are of the same form as the given equations expressing the y's in terms of the x's . . 163
127. Conditions which must exist before the ordinary rules of differentiation are allowable in the most elementary cases 163

MISCELLANEOUS PROBLEMS 164

CHAPTER VIII

CERTAIN FUNDAMENTAL CONCEPTIONS IN THE THEORY OF ANALYTIC FUNCTIONS

I. ANALYTIC DEPENDENCE; ALGEBRAIC DEPENDENCE

128. Rational functions of one or more variables. Functions defined through arithmetical operations. One-value functions. Infinite series and infinite products. Convergence 166
129. Uniform convergence 167
130. Region of convergence. Differentiation 167
131. Functions of several variables. Functions which behave like an integral rational function 168
132. Analytic dependence 169
133. Many-valued functions 169
134. Possibility of expressing many-valued functions through one-valued functions 169
135. An important theorem for the calculus of variations 171
136. The same theorem proved in a more symmetric manner 172
137. Application of this theorem and the definition of a *structure* of the $(n-m)$th kind in the realm of n quantities. Analytic continuation 174
138. Analytic structures defined in another manner. Structures of the first kind, second kind, etc. 175

II. ALGEBRAIC STRUCTURES IN TWO VARIABLES

139. *Algebraic structure* defined; *regular* point; *singular* point . . . 176
140, 141. Development of an algebraic function y in the neighborhood of a regular point. For such a point there is only one power series in x which when written for y makes identically zero the function which defines the algebraic structure 177

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III. Method of finding All Series for y which belong to a k-ply Singular Point

SECTION PAGE

142–143. Practical methods for the derivation of such series under the different conditions that may arise 180

144. Theorem of Stolz which serves as a check on the derivation of the above series 187

145. An example illustrating the general theory 187

INDEX 191

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THEORY OF MAXIMA AND MINIMA

CHAPTER I

FUNCTIONS OF ONE VARIABLE

I. ORDINARY MAXIMA AND MINIMA

1. A function $f(x)$ which is uniquely defined for all values of x in the interval (a, b) is said* to have a *greatest value* or a *maximum* for the value $x = x_0$, situated between a and b, if there is a positive quantity δ such that for all values of h between $-\delta$ and $+\delta$ the difference

$$[1] \qquad f(x_0+h) - f(x_0) \leqq 0$$

exists, which at the same time does not vanish for all these values of h. This function has a *smallest value* or a *minimum* if under the same conditions there exists the difference

$$[2] \qquad f(x_0+h) - f(x_0) \geqq 0,$$

which does not vanish for all values of h between $-\delta$ and $+\delta$.

A function may have several such maxima and minima which may be different from one another; it may have minima which are greater than maxima. (See the accompanying figure.) When the existence of complete derivatives in the entire interval under consideration is presupposed, the maxima and minima which may be derived are called *ordinary*; but when we have to do with functions whose derivatives exist only on definite points or with functions which have one-sided derivatives and the like, the maxima and minima may be called *extraordinary*. The discussion will be

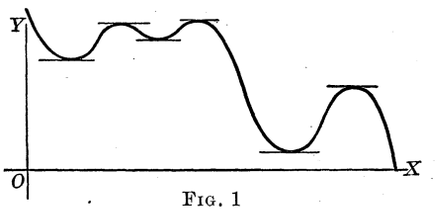

FIG. 1

* See Genocchi-Peano, *Calcolo differenziale e principii di calcolo integrale* (§ 131).

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restricted at first to ordinary maxima and minima. A maximum of $f(x)$ is called *proper* by Stolz (*Grundzüge der Differential- und Integralrechnung*, Vol. I, p. 199) if in the formula [1] only the sign $<$ stands; while we have a *proper* minimum if there stands only the sign $>$ in [2]. The maximum and minimum are *improper* if in formulas [1] and [2] the sign $=$ also appears, however small δ may be taken.

For example, $y = \left(x \sin \frac{1}{x}\right)^2$ has the value $+0$ when $x = \frac{1}{n\pi}$ for consecutive integral values of n, however large, and that is for intervals as small as we wish. Stolz and others * use the notation *extreme* or *extreme value* of a function to denote indifferently either the maximum or minimum of the function.

The maximum and minimum of a function defined as above are often denoted as *absolute* † maximum and minimum, since they depend upon the collectivity of the values of $f(x)$. Opposed to them appear the *relative* maximum or minimum, which enter if the independent variable x is subjected to a restriction so that h in the formulas [1] and [2] can take only restricted (and not arbitrary) positive and negative values.

2. If the function $f(x)$ has for $x = x_0$ a positive derivative $f'(x_0)$, the function is becoming larger on this position with increasing x, and its values are respectively smaller or greater than those of $f(x_0)$ according as x is smaller or greater than x_0. It is assumed that x lies sufficiently near x_0.

In this case the function $f(x)$ has for $x = x_0$ neither a maximum nor a minimum. Similar (*mutatis mutandis*) conclusions are drawn if $f'(x_0)$ is negative.

It follows that *if the function $f(x)$ has for $x = x_0$ a finite derivative that is different from zero, then on this position the function has neither a maximum nor a minimum.*

If then we exclude from the values of x those to which a definite derivative (different from zero) corresponds, there remain either

* *Extremer Werth* was used by R. Baltzer, *Elem. d. Math.*, Bd. I, Aufl. 5, S. 217; *Extremum* by P. du Bois-Reymond, *Math. Ann.*, Vol. XV, p. 564.

† The authors just cited, as also Peano, understand by the *absolute maximum* and *minimum* of a function in a given interval the *upper* and *lower limits* of the function in this interval, if such limits are reached. See also A. Mayer, *Leipz. Ber.* (1899), p. 122, and Lipschitz, *Analysis*, Vol. II, p. 306, and in particular Voss, *Encyklopädie der Math. Wiss.*, Bd. II, Theil I, Heft I, S. 80, who remarks upon the weak terminology of the subject.

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those positions on which the function has no derivative (finite or infinite) or those places on which it has a vanishing derivative.

These positions must be further examined if we wish to make ourselves sure of the existence or nonexistence of a maximum or minimum. No rule can be given for the cases where derivatives do not exist.

If we assume that the derivative is zero, the following criteria may be used: If $f(x)$ has the derivative $f'(x)$ in the interval $(x_0 - h \cdots x_0 + h)$, we have, in virtue of the Taylor formula,*

$$f(x) - f(x_0) = (x - x_0) f'(x_1),$$

where x_1 lies between x_0 and x. If $f'(x)$ becomes zero on the position $x = x_0$ in such a way that it is positive for $x < x_0$ and negative for $x > x_0$, then $(x - x_0) f'(x_1)$ is always negative, however $x (\neq x_0)$ be taken, and consequently it follows that $f(x) < f(x_0)$ for all values of x within the interval $x_0 - h$ to $x_0 + h$. The function will therefore be in this case a proper maximum for $x = x_0$. If, however, $f'(x)$ is negative for $x < x_0$ and positive for $x > x_0$, the product $(x - x_0) f'(x_1)$ is always positive, and the function will therefore be a proper minimum for $x = x_0$ within the interval in question.

If $f'(x)$ is zero, say, for values of x within the interval $x_0 \cdots x_0 + h$ or within the interval $x_0 - h \cdots x_0$, we have cases of improper extremes (maxima or minima). But if $f'(x)$ retains a constant sign in the neighborhood of $x = x_0$, then $(x - x_0) f'(x_1)$ changes its sign according as $x > x_0$ or $x < x_0$, and the function has neither a maximum nor a minimum.

It is thus seen that *the function $f(x)$ has on the position $x = x_0$ a maximum or a minimum according to the manner in which its derivative vanishes for $x = x_0$; that is, according as we pass from positive to negative values or from negative to positive values with increasing x. It has neither a maximum nor a minimum if the derivative does not change its sign.*†

* See Pierpont, *The Theory of Functions of Real Variables*, Vol. I, p. 248.

† Leibniz, Vol. V, pp. 220–226, is the first who made a distinction between maximum and minimum. See Maclaurin, *A Treatise of Fluxions* (1742), Vol. I, chap. ix, and Vol. II, p. 695; and also Cauchy, *Calc. différ.*, p. 63. With Leibniz, when $\frac{dy}{dx} = 0$, y is a *maximum* if the curve is concave towards the x-axis, a *minimum* if the curve is concave away from the x-axis.

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3. Instead of considering the sign of the derivative in the neighborhood of x_0, if we consider the sign of the second derivative for $x = x_0$ (when this second derivative is different from zero), we have the rule:

The function $f(x)$ has on the position $x = x_0$ for which $f'(x_0) = 0$ a maximum or a minimum according as $f''(x_0)$ is negative or positive. Infinite values are always included unless it is stated to the contrary.

In fact, if $f''(x_0) < 0$, then $f'(x)$ is a decreasing function, and since it is zero for $x = x_0$, it goes from positive to negative values; the inverse is the case if $f''(x_0) > 0$.

This rule leaves one in the lurch if $f''(x_0) = 0$.

If in general it is assumed that

$$f'(x_0) = 0, \quad f''(x_0) = 0, \quad \cdots, \quad f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0,$$

it follows from Taylor's formula that

$$f(x) - f(x_0) = \frac{(x - x_0)^n}{n!} f^{(n)}(x_1),$$

where x_1 is situated between x_0 and x.

As here $f^{(n)}(x)$ is assumed to be a continuous function, it retains a constant sign in the neighborhood of x_0. If n is odd, the factor $(x - x_0)^n$ changes sign according as $x > x_0$ or $x < x_0$. Consequently $f(x) - f(x_0)$ also changes its sign and $f(x_0)$ is neither a maximum nor a minimum. If n is even, the factor $(x - x_0)^n$ is positive and $f(x) - f(x_0)$ has always the sign of $f^{(n)}(x_1)$. It follows that $f(x_0)$ is a maximum or minimum according as $f^{(n)}(x_1)$ is negative or positive. We therefore have the theorem: *

If for $x = x_0$ the first and some of the following derivatives vanish, then $f(x_0)$ is or is not an extreme according as the first nonvanishing derivative is of even or odd order. If it is of even order, there is a maximum or a minimum according as the derivative in question is negative or positive.

* See Maclaurin, *A Treatise of Fluxions*, Vol. I, p. 226; Vol. II, p. 695; and also Lagrange, *Œuvres*, Vol. I (1759), p. 4. It was Maclaurin who first gave a correct method of distinguishing maxima from minima.

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4. The following may be regarded as a résumé of what has been given above: The function $f(x)$ is supposed to be uniquely defined for all values of x within an interval (a, b), and x_0 is a point of this interval. The function $f(x)$ is a proper maximum or a proper minimum for $x = x_0$ if we are able to find a positive number δ sufficiently small that the difference $f(x_0+h)-f(x_0)$ retains a constant sign when h varies from $-\delta$ to $+\delta$. If this difference is positive, the function $f(x)$ is smaller for $x = x_0$ than it is for the values of x neighboring x_0; it is then a *proper minimum.* On the contrary, when the difference $f(x_0+h)-f(x_0)$ is negative, the function is a *proper maximum* for $x = x_0$. If, furthermore, the sign $=$ enters in the cases just mentioned, however small δ be taken, we have an *improper minimum* or *maximum.*

When the function $f(x)$ admits a derivative for the value x_0 of the variable, this derivative must be zero. In fact, the two quotients

$$\frac{f(x_0+h)-f(x_0)}{h}, \qquad \frac{f(x_0-h)-f(x_0)}{-h}$$

which have here by hypothesis the same limit $f'(x_0)$, when h tends towards zero, are of different sign; it is necessary then that their common limit $f'(x_0)$ be zero.

Inversely, let x_0 be a root of the equation $f'(x) = 0$, situated between a and b, and taking the general case suppose that the first derivative which is not zero for $x = x_0$ is the derivative of the nth order and that this derivative is continuous in the neighborhood of the value x_0. The general formula of Taylor gives here, limiting it to the term of the nth degree,

$$f(x_0+h)-f(x_0) = \frac{h^n}{n!}f^{(n)}(x_0+\theta h) \qquad (0<\theta<1)$$
$$= \frac{h^n}{n!}[f^{(n)}(x_0)+\epsilon]$$

where ϵ is a quantity that is indefinitely small with h. Let δ be a positive number such that as x varies from $x_0-\delta$ to $x_0+\delta$ the absolute value of ϵ is smaller than $f^{(n)}(x_0)$, so that $f(x_0+h)-f(x_0)$ has the same sign as $\frac{h^n}{n!}f^{(n)}(x_0)$. If n is odd, we note that this difference changes sign with h; there is then neither a maximum

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nor a minimum for $x = x_0$. If n is even, $f(x_0+h)-f(x_0)$ has the same sign as $f^{(n)}(x_0)$ whether h be positive or negative; the function is a *minimum* if $f^{(n)}(x_0)$ is positive, and a *maximum* if $f^{(n)}(x_0)$ is negative. It follows that for the function to be a maximum or minimum for $x = x_0$ it is necessary and sufficient that the first derivative which vanishes for $x = x_0$ be of even order.*

In geometric language the preceding conditions denote that the tangent to the curve $y = f(x)$ at the point P_0 is parallel to OX and is *not* an inflectional tangent (see Figs. 2–5).

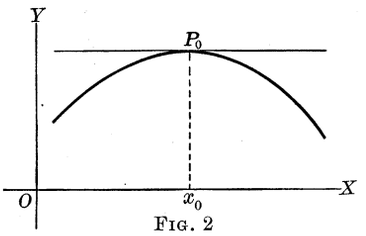

FIG. 2

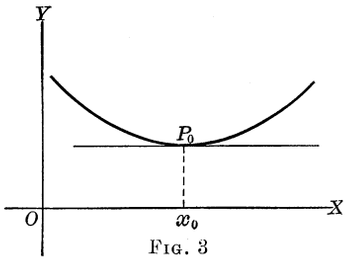

FIG. 3

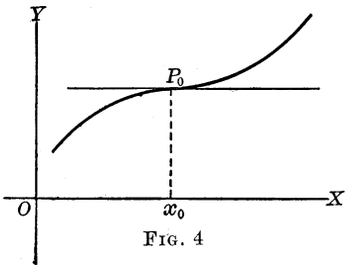

FIG. 4

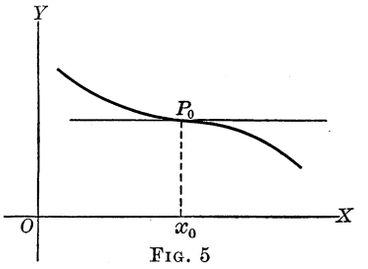

FIG. 5

II. EXTRAORDINARY MAXIMA AND MINIMA

A. FUNCTIONS WHICH HAVE DERIVATIVES ONLY ON DEFINITE POSITIONS

5. Let the function $y = f(x)$ be uniquely defined for all values of x between $x_0 - \delta$ and $x_0 + \delta$ and suppose that it is continuous for $x = x_0$. If the expressions

$$\frac{f(x_0+h)-f(x^0)}{h} \quad \text{and} \quad \frac{f(x_0-h)-f(x_0)}{-h} \qquad (h>0)$$

* See Goursat, *Cours D'Analyse*, Vol. I, pp. 108 et seq. I shall refer hereafter to this work by the *name* of the author, and by Peano and Stolz I shall designate the works, cited above, of these two mathematicians.

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have limiting values when $\lim h = +0$, each of these expressions is called a *one-sided* differential quotient,* the first the *right-hand*, and the second the *left-hand*, differential quotient of $f(x)$ with regard to x for the value $x = x_0$.

If the two one-sided differential quotients of $f(x)$ are equal to each other for $x = x_0$, their common value is called the *complete* differential quotient of $f(x)$ with respect to x for $x = x_0$.

If next it is assumed that the one-valued function $f(x)$ is continuous for all values of the interval (a, b) and has at least one-sided differential quotients, the differential calculus offers a method for the determination of the maxima and minima. For if $f(x_0)$ is such an extreme of $f(x)$, the quotient

$$\frac{f(x_0+h)-f(x_0)}{h}$$

must necessarily either vanish or change sign with h.

It may therefore be concluded, as in § 2, that the complete differential quotient $f'(x)$ must be zero, and that if the right-hand and left-hand differential quotients of $f(x)$ are different at the point $x = x_0$, they cannot have the same sign. These requirements are under the existing conditions necessary that $f(x_0)$ be an extreme of $f(x)$; however, as it will be seen in the following, they are not always sufficient.

6. *Criteria as to whether a root $x = x_0$ of the equation $f'(x) = 0$ offers an extreme of the function $f(x)$.*†

THEOREM I. If $f'(x)$ vanishes for $x = x_0$, and if a positive quantity δ may be so chosen that $f(x)$ has complete differential quotients in the interval $(x_0 - \delta \cdots x_0 + \delta)$, and if $f'(x)$ changes sign neither in the interval $(x_0 - \delta \cdots x_0)$ nor in the interval $(x_0 \cdots x_0 + \delta)$ and also does *not* remain invariably zero in either of these intervals, then $f(x_0)$ is or is not a proper extreme of $f(x)$ according as the sign of $f(x)$ in the first interval is different from or the same as it is in the second interval; and furthermore, in the first case $f(x_0)$ is a *maximum* or a *minimum* of

* See P. du Bois-Reymond, *Math. Ann.*, Vol. XVI, p. 120; see also Pierpont, *The Theory of Functions of Real Variables*, Vol. I, p. 223.

† See Cauchy, *Calc. différ.*, *Lesson 7*, and see in particular Stolz, pp. 201–210.

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$f(x)$ according as $f'(x)$ on its passage through zero, as x with increasing values passes through x_0, changes from the sign + to the sign − or from the sign − to the sign +.

This theorem is stated at the end of § 3 and there proved; and as also indicated in that section, the inconvenience arising due to the consideration of the sign $f'(x)$ may be obviated if the function $f(x)$ has a complete second differential quotient for $x = x_0$. This leads to the following theorem:

THEOREM II. If under the conditions assumed in Theorem I the function $f(x)$ has for $x = x_0$ a complete second differential coefficient $f''(x_0)$ which is not zero, then $f(x_0)$ is a proper extreme of $f(x)$, being a maximum or minimum according as $f''(x_0)$ is negative or positive.

Due to the definition of a complete second derivative it follows that

$$\frac{f'(x+h)-f'(x)}{h} = f''(x) + R(h),$$

where $R(h)$ is a quantity that becomes indefinitely small with h.

If here the existence of a second derivative of $f(x)$ is assumed *only* for $x = x_0$, then, since $f'(x_0) = 0$, there corresponds to every positive quantity ϵ another quantity δ such that, if $-\delta < h < \delta$, we have

$$f''(x_0) + \epsilon > \frac{f'(x_0+h)}{h} > f''(x_0) - \epsilon.$$

If, say, $f''(x_0)$ is *positive*, it follows at once that there must be a positive quantity δ such that for $-\delta < h < \delta$ we have

$$[3] \qquad \frac{1}{h} f'(x_0+h) > 0.$$

Hence $f'(x_0+h)$ must be negative when h is negative and positive when h is positive, so that on passing through zero (i.e., when $x = x_0$), $f'(x_0+h)$ passes with increasing x from a negative value to a positive value. Accordingly, in virtue of Theorem I, $f(x_0)$ is a proper minimum.

The above theorem becomes the one given in § 3 if it is assumed that there is an interval including the value $x = x_0$ such that for all points within it a second differential quotient of $f(x)$

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exists, and if it is further assumed that $f''(x)$ is continuous at least at the point $x = x_0$.

THEOREM III. If, furthermore, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then $f(x_0)$ is *not* an extreme of $f(x)$.

For since here

$$f'(x_0 + h) - f'(x_0) = \frac{h^2}{2} f'''(x_0) + \cdots,$$

it is seen that as $f'(x)$ passes through the value $f'(x_0) = 0$, it does *not* change sign, and consequently $f(x_0)$ is *not* an extreme of $f(x)$.

The two preceding theorems are special cases of the two following:

THEOREM IV. If for the value $x = x_0$ we have

$$f'(x_0) = 0, \quad f''(x_0) = 0, \quad \cdots, \quad f^{(2k-1)}(x_0) = 0, \quad f^{(2k)}(x_0) \neq 0,$$

then $f(x_0)$ is a proper extreme of $f(x)$, being a minimum or maximum according as $f^{(2k)}(x_0)$ is positive or negative.

For, owing to the supposed existence of the first $2k$ differential quotients, there is an interval $x_0 - \delta \cdots x_0 + \delta$ throughout which the differential quotients $f'(x), f''(x), \cdots, f^{(2k-1)}(x)$ not only exist but are also continuous functions of x. Accordingly, in virtue of Taylor's formula, we have

$$[4] \quad f(x_0 + h) - f(x_0) = \frac{h^{2k-1}}{(2k-1)!} f^{(2k-1)}(x_0 + \theta h), \quad (0 < \theta < 1)$$

which formula is true for all values of h such that $-\delta < h < \delta$. Owing to the existence of $f^{(2k)}(x_0)$, as in the case of formula [3] above, it is seen that for values of h such that $-\delta < h < \delta$ we have

$$\frac{1}{h} f^{(2k-1)}(x_0 + h) > 0 \quad \text{or} \quad < 0$$

according as $f^{(2k)}(x_0)$ is positive or negative.

If, then, $f^{(2k)}(x_0)$ is, for example, positive, it is clear that $f^{(2k)}(x_0 + h)$ is negative for values of h in the interval $-\delta \cdots 0$ and positive for values of h in the interval $0 \cdots \delta$.

It follows from [4] that the difference $f(x_0 + h) - f(x_0)$ for all values of h within the interval $-\delta \cdots + \delta$ (excepting $h = 0$) is invariably $+$ or $-$ according as $f^{(2k)}(x_0)$ is $+$ or $-$, and correspondingly we have respectively a proper minimum or a proper maximum.

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If it is further supposed that $f^{(2k)}(x)$ exists for all values of x in the interval $x_0 - \delta \cdots x_0 + \delta$ and that $f^{(2k)}(x)$ is a continuous function at least at $x = x_0$, then, as in § 3, due to Taylor's expansion we have

$$[5] \qquad f(x_0 + h) - f(x_0) = \frac{h^{2k}}{(2k)!} f^{(2k)}(x_0 + \theta h), \qquad (0 < \theta < 1)$$

from which the theorem is obvious.

In exactly the same way we may prove

THEOREM V. If for $x = x_0$ the $2k$ first differential quotients of $f(x)$, viz., $f'(x_0)$, $f''(x_0)$, $\cdots$, $f^{(2k)}(x_0)$, vanish, and if $f^{(2k+1)}(x_0) \neq 0$, then $f(x_0)$ is not an extreme of $f(x)$.

REMARK. In the case that $x = x_0$ causes every differential quotient of $f(x)$ to vanish, we cannot determine by means of Theorems II, III, and IV whether $f(x)$ is an extreme or not. We must then apply Theorem I. For example, it is seen that $x = 0$ is a minimum of $f(x) = e^{-\frac{1}{x^2}}$.

THEOREM V^a. If the given function $f(x)$ can be developed in the neighborhood of the point x_0 in a series in integral positive powers of $x - x_0 = h$ so that

$$f(x) = f(x_0 + x - x_0) = c_m h^m + c_{m+1} h^{m+1} + \cdots, \qquad (c_m \neq 0)$$

then $f(x_0)$ is *not* an extreme of $f(x)$ or *is* an extreme of $f(x)$ according as m is odd or even; and $f(x_0)$ is a maximum or minimum according as c_m is negative or positive.

For here $$f'(x_0) = 0 = f''(x_0) = \cdots = f^{(m-1)}(x_0),$$

while $$f^{(m)}(x_0) = m!\, c_m.$$

This theorem may be proved directly by means of the property of series. For under the given assumptions corresponding to every quantity $\epsilon > 0$, we may choose another quantity $\delta > 0$, such that

$$-\epsilon < c_{m+1} h + c_{m+2} h^2 + \cdots < \epsilon.$$

If m is a positive integer and $c_m < 0$, say, and if $|\lambda| < \delta$,

then $$f(x_0 + h) - f(x_0) < h^m (c_m + \epsilon);$$

and as ϵ may be taken such that $\epsilon < -c_m$, the expression on the right-hand side is always *negative*, so that there is a maximum of $f(x)$ at $x = x_0$. Similarly, we may prove the remaining part of V^a.

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E. Functions which have only One-Sided Differential Quotients of a Certain Order for a Value $x = x_0$*

7. Theorem VI. If the continuous function $f(x)$ has for $x = x_0$ one-sided differential quotients of the first order and of opposite sign (including $+\infty$ and $-\infty$), then $f(x_0)$ is a proper extreme, being a maximum or minimum according as the right-hand differential quotient of $f(x)$ is negative or positive.

For if, say, the left-hand differential quotient is positive, the right-hand one being negative, then there exists a positive quantity δ such that according as $-\delta < h < 0$ or $0 < h < \delta$, we have

$$\frac{f(x_0+h)-f(x_0)}{h} > 0 \quad \text{or} \quad < 0.$$

It follows that $f(x_0+h)-f(x_0) < 0$ for all values of h that are situated within the interval $-\delta \cdots +\delta$. Hence $f(x_0)$ is a proper maximum.

Theorem VII. If for $x = x_0$ the $2k$ first differential quotients of $f(x)$, viz., $f'(x), f''(x), \cdots, f^{(2k)}(x)$, vanish, and if $f^{(2k)}(x)$ has for $x = x_0$ one-sided differential quotients of contrary sign ($+\infty$ and $-\infty$ included), then the value $f(x_0)$ forms a proper extreme of $f(x)$, being a maximum or a minimum according as the right-hand differential quotient is negative or positive.

If, for example, the left-hand differential quotient is positive, while the right-hand is negative, that is,

$$\frac{f^{(2k)}(x_0-h)-f^{(2k)}(x_0)}{-h} > 0 \quad \text{and} \quad \frac{f^{(2k)}(x_0+h)-f^{(2k)}(x_0)}{h} < 0,$$

we note, since $f^{(2k)}(x_0) = 0$, that $f^{(2k)}(x_0+h) < 0$ for all values of h within the interval $-\delta \cdots +\delta$ (the value $h = 0$ excepted). Hence from formula [5], viz.,

$$f(x_0+h)-f(x_0) = \frac{h^{2k}}{(2k)!} f^{(2k)}(x_0+\theta h),$$

it is seen at once that $f(x_0)$ is a proper maximum.

Theorem VIII. If for $x = x_0$ the $2k-1$ first differential quotients vanish, viz., $f'(x_0) = f''(x_0) = \cdots = f^{(2k-1)} = 0$, and if

* Stolz, p. 206; see also Pascal, *Exercici*, etc., pp. 215–222. 1895.

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$f^{(2k-1)}(x)$ has for $x = x_0$ one-sided differential quotients of opposite sign ($+\infty$ and $-\infty$ included), then $f(x_0)$ does *not* form an extreme of $f(x)$. If, however, these differential quotients are both positive or both negative, then $f(x_0)$ is a proper minimum or maximum of $f(x)$. This theorem follows from [4] in the same manner as the preceding one did from [5].

Example. If $f(x) = x^\mu$ $(x \geqq 0)$ and $f(x) = (-x)^\mu$ $(x < 0)$, show by means of Theorem VI that there is a proper minimum at $x = 0$ if μ lies between 0 and $+1$. Verify the same result when μ lies between $2k$ and $2k+1$ by making use of Theorem VII; and by using Theorem IV show that $f(x)$ is a proper minimum when μ is situated between $2k-1$ and $2k$.

C. Upper and Lower Limits of a One-Valued Function which is Continuous for Values of the Argument within a Definite Interval

8. If the function $f(x)$ is continuous and uniquely defined in the definite interval (a, b), there exist the greatest and the least value* in the interval in question, which are known as the *upper* and *lower* limits of the function in this interval; and, further, the function reaches these limits; that is, if these limits are denoted by g and k, then there is at least one value c of x within the interval $a \cdots b$ for which the function is equal to g, and at least one value d within the same interval for which the same function is equal to k.

But if the interval within which x varies is indefinitely large, (a, ∞) or $(-\infty, b)$ or $(-\infty, +\infty)$, the function need not have a maximum or a minimum, and also it need not have an upper or a lower limit. This is illustrated in the following examples.† (See also § 96.)

* Proofs of this and the following statements are found, for example, in Harkness and Morley, *Intr. to Analytic Functions*, §§ 46, 50; E. B. Wilson, *Advanced Calculus*, §§ 19–25. See Peano, Theorem IV, § 21, and also Dini, *Fundamenti per la teorica delle funzioni di variabili reali* (German translation by Lüroth and Schepp, §§ 36, 47). These proofs are founded upon Weierstrass's lectures, which, in turn, are founded upon the work of Bolzano, *Abh. d. Böhmischen Gesellsch. der Wiss.*, Vol. V, p. 17.

† Peano, § 132.

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Example 1. *Divide a number into two parts so that their product is a maximum.* (Cf. Ex. 6 at end of § 10.)

Let a be the given number, x and $a-x$ the two summands, and $y=(a-x)x$ their product. If we consider x as variable, we have $y'=a-2x$, which becomes zero for $x=\frac{a}{2}$. We further have $y''=-2$, so that the function y has a maximum for $x=\frac{a}{2}$; that is, when both parts are equal, this value being $y=\frac{a^2}{4}$.

Since, however, the derivative y' is positive for $x<\frac{a}{2}$ and negative for $x>\frac{a}{2}$, it follows that the function increases in the interval $\left(-\infty, \frac{a}{2}\right)$ and decreases in the interval $\left(\frac{a}{2}, +\infty\right)$. The function has neither an upper nor a lower limit.

Example 2. $y=x^x$. $\quad (x>0)$

Through differentiation we have $y'=x^x(1+\log x)$. The first factor is never zero and is always *positive*. The second factor becomes zero when $\log x=-1$ or $x=\frac{1}{e}$. The derivative passes therefore from negative $\left(\text{for } x<\frac{1}{e}\right)$ to positive values $\left(\text{for } x>\frac{1}{e}\right)$. The function has a minimum for $x=\frac{1}{e}=0.36788\cdots$, which is $y=0.676411\cdots$. This is also the lower limit which the function takes in the interval $(0, \infty)$. The function does *not* have either a maximum or an upper limit.

Example 3. $y=x^{\frac{2}{3}}$, $\quad y'=\frac{2}{3}x^{-\frac{1}{3}}$.

The derivative is zero for no finite value of x, but is infinite for $x=0$. For this value y becomes zero, and the function will have at this point both a minimum and a lower limit with respect to the interval $(-\infty, +\infty)$; for all the values of x cause the function to be greater than zero. The function has neither a maximum nor an upper limit.

9. If we add to the postulates already made in the previous article regarding $f(x)$ that it must have a complete differential quotient for all values of x between a and b, then $f'(x)$ vanishes for every value of x *between* a and b to which one of the values g or k of the function belongs. If, however, $f(a)=g$, say, then possibly $f(a)$ is only a one-sided maximum of $f(x)$, and consequently $f'(a)$ is *not* necessarily zero. This must be borne in mind as we proceed with the problem of determining the numbers g and k. This is

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explained by the simple example (see *Liouville's Journal*, First Series, Vol. VII, p. 163):

In the plane of a circle which is described about the point O as center with radius r, let there be given an arbitrary point A which is different from O. Determine the upper and lower limits of the distances of the point A from a point M of the circumference.

Let the positive X-axis be taken passing through O, and standing perpendicular to it through O is erected the Y-axis. The equation of the circle is then $x^2+y^2=r^2$; while

$$\overline{AM}^2=(a-x)^2+y^2=r^2+a^2-2ax. \qquad (\alpha)$$

As M passes over all points of the circumference, x takes the values in the interval $-r \cdots +r$. The linear function (α) decreases with increasing values of x, its differential quotient being a negative constant equal to $-2a$. Consequently those values of x to which the upper and lower limits of $\overline{AM}^2$ belong, fall on the end-points of the interval $-r \cdots +r$. It is seen that $-r$ corresponds to the upper limit and $+r$ to the lower limit, giving us as upper and lower limits respectively $|a+r|$ and $|a-r|$.

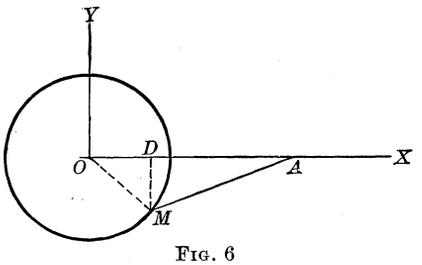

FIG. 6

10. Suppose next that the function $f(x)$ is discontinuous at least on an end-point of the arbitrary interval (a, b); for example, suppose that the function is not defined at such a point. If this is the case only for the $\lim x=a$, then in the derivation of the upper and the lower limit we must consider in particular the value of $f(x)$ for the $\lim x=a+0$. The following examples will make clear the method of procedure (see Stolz, p. 210).

Example 1. Consider the function $y=\dfrac{1}{\log x}$ for values of x such that $0<x<1$. It is seen that y is negative and decreases with increasing values of x. For when $\lim x=+0$, then $\lim y=-0$;

and when $\lim x=1-0$, then $\lim y=-\infty$.

Thus the upper limit of y in this interval is zero, while the lower limit is $-\infty$, although neither is reached.

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Example 2. $y = (1-x)\sin\frac{\pi}{x}$. $\quad (0 < x \leqq 1)$

For these values of x we have always $|y| < 1$.

If we consider only values of x such as $x = \frac{2}{4n+1}$ (where n is an integer), we have

$$y = \left(1 - \frac{2}{4n+1}\right)\sin\left(\frac{\pi}{2}(4n+1)\right) = 1 - \frac{2}{4n+1}.$$

Hence when $n = +\infty$, the upper limit of y is $+1$.

By writing $x = \frac{2}{4n-1}$, it is seen that the lower limit is -1. Neither the upper nor the lower limit of y is reached, although in either case they are finite.

PROBLEMS

1. Determine the maxima and minima and the upper and lower limits of

(a) $y = \frac{\log x}{x}$. $\quad (x > 0)$

(b) $y = a\cos x + b\sin x$.

(c) $y = a + x^{\frac{2}{3}}$. (Pierpont, p. 320.)

(d) $y = 1 - x^{\frac{5}{3}}$. (Maclaurin, Vol. II, p. 720.)

(e) $x^2 \sin\frac{1}{x}$. (The function has a discontinuous derivative for $x = 0$.)

(f) $y = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$.

(g) $y = x^2 - e^{\frac{1}{x^2}}$.

(h) $y = e^{-\frac{1}{x^2}}$.

(i) $y = xe^{-\frac{1}{x^2}}$. (There is no extreme on the position $x = 0$.)

2. Show that the function $\begin{cases} f(x) = x\sin\frac{\pi}{x} & (x \neq 0) \\ f(0) = 0 \end{cases}$

has an infinite number of maxima and minima within the interval $(-1 \cdots +1)$.

3. When is $mp + nq$ a minimum, where $p = \sqrt{c^2 + y^2}$, $q = \sqrt{k^2 + (h-y)^2}$? (Leibniz, 1682.)

4. "Invenire cylindrum maximi ambitus in data sphaera." (Fermat, *Œuvres*, Vol. I, p. 167. 1642.)

5. Find the area of the greatest parabola which may be cut from a given cone.

6. $x(x-a)$ has its greatest value when $x = \frac{a}{2}$. (Euclid, Book VI, Prop. 27.) Cantor (*Geschichte der Math.*, Vol. I, p. 228) says that this is the first example of a maximum in the history of mathematics.

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7. On a given line AB are two fixed points P_1 and P_2. Determine a third point so that $\frac{AP \cdot PB}{P_1P \cdot PP_2}$ is a minimum. (Pappus, Book VII, Prop. 61, and Fermat, *Œuvres*, Vol. I, p. 140.)

8. Of all sections which pass through the vertex of a cone, determine the one of greatest area. (Severus.)

9. The number a is to be divided into two parts, such that their product multiplied into their difference shall be a maximum. (Tartaglia, *General Trattato*, Part 5, fol. 88.)

10. A ten-foot pole hangs vertically so that its lower end is four feet from the floor. Find the point on the floor from which the pole subtends the greatest angle. (Regiomontanus. 1471.)

11. The curve $y = x^2 - \frac{1}{\log x}$ has no *minimum*. (Euler, *Differentialrechnung*, Vol. III, p. 744.)

12. Two points P_1 and P_2 not on the straight line AB are given. Find a point P on AB such that $\overline{PP}_1^2 + \overline{PP}_2^2$ is a minimum. (Solved by Huygens possibly about 1673. See Huygens, *Opera Varia*, pp. 490 et seq. Note the letters of De Sluse.)

13. Derive the greatest rectangle that can be described in, and having one of its sides, upon the base of a given triangle. (Simpson, *Elements of Plane Geometry* (1747), pp. 106 et seq. In this work are also found numerous problems that have to do with areas, volumes, etc.)

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CHAPTER II

FUNCTIONS OF SEVERAL VARIABLES

I. ORDINARY MAXIMA AND MINIMA

Preliminary Remarks

11. We say that the function $u = f(x_1, x_2, \cdots, x_n)$ becomes a maximum or minimum on the position $(a_1, a_2, \cdots, a_n)$ if for a sufficiently small region about $(a_1, a_2, \cdots, a_n)$ we have

$$f(a_1, a_2, \cdots, a_n) \geqq f(x_1, x_2, \cdots, x_n)$$

or

$$f(a_1, a_2, \cdots, a_n) \leqq f(x_1, x_2, \cdots, x_n).$$

These extremes are *proper* or *improper* according as the sign $=$ does *not*, or does enter.

As in § 1, it is assumed here that the function has definite partial derivatives which are continuous within the region in question with regard to each of the variables; and the extremes which may be derived we shall call *ordinary*. If the partial derivatives do not have such derivatives, the extremes may be called extraordinary. Such extremes in their generality we shall not attempt to consider. Another class of extraordinary extremes is mentioned in § 13, and is later treated in its generality for the case of functions of two variables (§§ 20 et seq.).

12. Consider the function of one variable x_1, viz., $f(x_1, a_1, \cdots, a_n)$. If the function u of the preceding article is a maximum or minimum for $x_1 = a_1, \cdots, x_n = a_n$, then $f(x_1, a_1, \cdots, a_n)$ will be a maximum or minimum for $x_1 = a_1$. Hence (see § 2) the derivative $f'_{x_1}(x_1, a_2, \cdots, a_n)$ must be zero. Similar conclusions may be made for the other variables in u.

It follows that *if* $u = f(x_1, \cdots, x_n)$ *has an extreme on the position* $(a_1, a_2, \cdots, a_n)$, *the first partial derivatives of* u *must be zero.*

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(See Euler, *Calc. diff.* (1755), p. 645; and Lagrange, *Théorie des Fonctions*, Vol. II, No. 51.)

Write* next $x_1 = a_1 + h_1 t, \cdots, x_n = a_n + h_n t$, and put

$$F(t) = f(a_1 + h_1 t, \cdots, a_n + h_n t).$$

If $u = f(x_1, \cdots, x_n)$ is an extreme on the position $(a_1, \cdots, a_n)$, then $F(t)$ is an extreme on the position $t = 0$.

Since by hypothesis the derivatives of u are continuous, it follows also that the same is true of $F(t)$.

We consequently may write

$$F'(t) = f'_{x_1}(x_1, x_2, \cdots, x_n)h_1 + f'_{x_2}(x_1, x_2, \cdots, x_n)h_2 + \cdots + f'_{x_n}(x_1, x_2, \cdots, x_n)h_n.$$

It follows from § 2 that

$$F'(0) = f'_{a_1}(a_1, \cdots, a_n)h_1 + \cdots + f'_{a_n}(a_1, \cdots, a_n)h_n = 0,$$

whatever be the values of $h_1, h_2, \cdots, h_n$.

We therefore have

$$f'_{a_1}(a_1, \cdots, a_n) = 0, \cdots, f'_{a_n}(a_1, \cdots, a_n) = 0,$$

as was just seen.

We further have

$$F''(t) = f''_{x_1x_1}(x_1, x_2, \cdots, x_n)h_1^2 + f''_{x_2x_2}(x_1, x_2, \cdots, x_n)h_2^2 + \cdots + 2f''_{x_1x_2}(x_1, x_2, \cdots, x_n)h_1h_2 + \cdots.$$

If u is to be an extreme for the position under consideration, then $F(t)$ must be an extreme for $t = 0$, so that for a maximum we must have (§ 3) $F''(0) \leqq 0$, and for a minimum $F''(0) \geqq 0$, whatever be the values of $h_1, h_2, \cdots, h_n$. If for the time being we omit the sign $=$ from the two expressions just written, we have the theorem:

In order that the function u be an extreme at the position $(a_1, \cdots, a_n)$ for which the first derivatives vanish, it is necessary that the following homogeneous function of the second degree in $h_1, \cdots, h_n$, viz.,

$$F''(0) = f''_{a_1a_1}(a_1, a_2, \cdots, a_n)h_1^2 + f''_{a_2a_2}(a_1, a_2, \cdots, a_n)h_2^2 + \cdots + 2f''_{a_1a_2}(a_1, a_2, \cdots, a_n)h_1h_2 + \cdots,$$

* See also Peano, § 134.

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assume only positive or only negative values, whatever be the values of $h_1, \cdots, h_n$, *except when these quantities are all simultaneously zero.*

13. We distinguish three kinds of integral functions of the second degree, or as they are usually called, *quadratic forms*,* viz.,

I. *Definite forms*, which with real values of the variables have always the same sign, that is, only positive values or only negative values, and are only zero when the variables are all zero.

II. *Semi-definite forms*,† which always have the same sign, but which vanish also for other values of the variables that are *not* all zero.

III. *Indefinite forms*, which with real values of the variables can become both positive and negative, and that too for values of the variables whose absolute values do not exceed an arbitrary small quantity.

The theorem of the preceding section may be written as follows:

If for $x_1 = a_1, \cdots, x_n = a_n$ the first partial derivatives of the function $u = f(x_1, \cdots, x_n)$ vanish, and if in the Taylor development‡ for $f(x_1 + h_1, \cdots, x_n + h_n)$ the term which is a homogeneous function of the second degree in $h_1, \cdots, h_n$ is an indefinite form, then u on the position $(a_1, \cdots, a_n)$ has neither a maximum nor a minimum value. If, on the other hand, that term is a *positive* definite form, then u is a *minimum*, and if it is a *negative* definite form, u is a *maximum*.

The case where the form is semi-definite is included under the *extraordinary* extremes, and we shall consider it later (§§ 20 et seq.).

14. Next is given a criterion to determine whether a given quadratic form $\phi(h_1, \cdots, h_n)$ is a *positive* definite quadratic form.

If ϕ depends only upon one variable h_1, we shall have $\phi = Ah_1^2$, and this is positive when and only when A is positive.

If ϕ depends upon two variables h_1 and h_2, we shall have

$$\phi = Ah_1^2 + 2Bh_1h_2 + Ch_2^2.$$

* See Gauss, *Disq. Arithm.*, p. 271.

† So called, for example, by Scheeffer, *Math. Ann.*, Vol. XXXV, p. 555. Gergonne, *Gerg. Ann.*, Vol. XX (1830), p. 331, called attention in particular to this case.

‡ This development is found in full in § 50.

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If here ϕ is a positive definite form, it follows that for $h_2 = 0$, $h_1 \neq 0$, then $\phi = Ah_1^2$, and consequently A must be *positive.* We may also write ϕ in the form

$$\phi = \frac{1}{A}[(Ah_1 + Bh_2)^2 + (AC - B^2)h_2^2].$$

If in this expression we give to h_1 and h_2 such values that $Ah_1 + Bh_2 = 0$, it is seen that ϕ takes the form $\phi = \frac{1}{A}(AC - B^2)h_2^2$. We must therefore have $AC - B^2 > 0$.

The conditions $A > 0$ and $AC - B^2 > 0$ are not only necessary, but they are also sufficient that ϕ be a definite quadratic form. In fact, if $h_2 \neq 0$, we have $(AC - B^2)h_2^2 > 0$ and $(Ah_1 + Bh_2)^2 \geqq 0$, and consequently the sum of these two expressions, and also ϕ, is positive.

If, in general, ϕ depends upon several variables h_1, h_2, h_3, $\cdots$, we may write

$$\phi = Ah_1^2 + 2Bh_1 + C,$$

where A is a constant, B a form of the first degree in h_2, h_3, $\cdots$, and C a quadratic form in h_2, h_3, $\cdots$.

If h_2, h_3, $\cdots$ are all zero, but $h_1 \neq 0$, we will have B and C zero and $\phi = Ah_1^2$. We must therefore have $A > 0$, if ϕ is to be a positive definite form.

The form may be written

$$\phi = \frac{1}{A}[(Ah_1 + B)^2 + (AC - B^2)],$$

where $AC - B^2$ is a quadratic form of h_2, h_3, $\cdots$. The quantity h_1 may be determined so that $Ah_1 + B = 0$ with the result that

$$\phi = \frac{1}{A}(AC - B^2).$$

Hence the expression $AC - B^2$ must be positive and different from zero.

Next write $AC - B^2 = \phi_1(h_2, h_3, \cdots)$, where ϕ_1 is a quadratic form in h_2, h_3, $\cdots$ which is always positive and different from zero except when all the variables vanish.

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It follows that the necessary conditions that ϕ be a definite positive form are (1) that A be greater than 0 and (2) that $AC - B^2$ be a positive definite form in the variables $h_2, h_3, \cdots$.

These conditions are also sufficient; for if we give to h_1 an arbitrary value and to $h_2, h_3, \cdots$ arbitrary values which are not all zero, then of the two summands into which ϕ is distributed, the first is positive or zero, while the second is positive. It follows that ϕ is *positive*. On the other hand, if we give to $h_2, h_3, \cdots$ simultaneously the value zero, then h_1 must be different from zero, and from $\phi = Ah_1^2$ it is seen that A must be positive. In this way the determination of the question whether a quadratic form is definite and positive is reduced to the determination of the same question in the case of another quadratic form of fewer variables. If then the process is continued, we come to the forms in one or two variables already considered. This subject is further considered in §§ 53 et seq.

To determine whether a quadratic form ϕ is definite and *negative*, we have to determine whether $-\phi$ is definite and positive. (See Peano, § 137.)

II. RELATIVE MAXIMA AND MINIMA

15. To introduce the theory, we shall consider here a simple case involving only three variables. Let it be required to determine the extremes of the function

$$u = F(x, y, z),$$

where the variables x, y, z are restricted. Suppose, for example, that they are connected by the equation

$$f(x, y, z) = 0.$$

If from the latter equation z is expressed as a function of x and y, and if this value is substituted in the first equation, we shall have u expressed as a function of x and y. The values x, y which make u a maximum or minimum cause the total derivative du to vanish for all values dx and dy.

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We have $$du = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{dz}dz,$$

where dz denotes the differential of z, which is defined through the equation

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0.$$

If this last equation is multiplied by the indeterminate quantity $-\lambda$ and added to the equation $du = 0$, we have

$$\left(\frac{\partial F}{\partial x} - \lambda\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} - \lambda\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial F}{\partial z} - \lambda\frac{\partial f}{\partial z}\right)dz = 0.$$

If in this equation we choose λ so that the coefficient of dz vanishes, then corresponding to the maxima and minima values of u the coefficients of dx and dy must also be zero, and we thus have the equations

$$\frac{\partial F}{\partial x} - \lambda\frac{\partial f}{\partial x} = 0, \quad \frac{\partial F}{\partial y} - \lambda\frac{\partial f}{\partial y} = 0, \quad \frac{\partial F}{\partial z} - \lambda\frac{\partial f}{\partial z} = 0.$$

It is evident that we have these expressions which are symmetric with regard to the three variables if we form the three partial derivatives of $F - \lambda f$, where λ is an indeterminate quantity, and then put the resulting expressions equal to zero.

These three equations, together with the two equations $f = 0$ and $u = F$, determine the unknown quantities λ, x, y, z, u, which correspond to the values of u for which there exist maxima and minima values.

We may proceed in the same manner with an arbitrary number of variables and equations of condition. (See Lagrange, *Théorie des Fonctions*, p. 268.)

PROBLEMS

1. Find the minimum value of u, where

$$u = x^2 + y^2 + z^2 + \cdots,$$

and where $x, y, z, \cdots$ are connected by the equation

$$ax + by + cz + \cdots = k.$$

2. If $x_1 + x_2 + \cdots + x_n = a$, show that

$$x_1^k + x_2^k + \cdots + k_n^k$$

is a minimum when $x_1 = x_2 = \cdots = x_n$. (Maclaurin.)

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CHAPTER III

FUNCTIONS OF TWO VARIABLES

I. ORDINARY EXTREMES

16. Let $z=f(x, y)$ be a continuous function of the two variables x and y when the point P with coördinates (x, y) remains within the interior of an area Ω which is limited by a contour C. We say that this function $f(x, y)$ is a *minimum* for a point (x_0, y_0) of the area Ω when we can find a positive quantity δ such that we have

$$\Delta=f(x_0+h, y_0+k)-f(x_0, y_0)\geqq 0 \qquad (i)$$

for all systems of values of the increments h and k that are less than δ in absolute value. The maximum is defined in a similar manner.*

If we exclude the sign $=$ in the expressions $\Delta\geqq 0$ or $\Delta\leqq 0$, the extremes are said to be *proper* (cf. § 1); but if the equality $\Delta=0$ exists for certain values of h and k that are less than δ in absolute value, however small δ be taken, we have *improper* extremes. For example, in the case of the surface represented by the equation $z=f(x, y)$, the axis Oz being vertical, a proper maximum corresponds to an isolated summit, but if these summits form a line on the surface, this line will be a line of improper maxima. Consider, for example, the lines generated by revolving the extremes of a plane curve about the Ox-axis.

If in the expression (i) we regard y as constant and equal to y_0, then z becomes a function of one variable x and (§ 2) the difference

$$f(x_0+h, y_0)-f(x_0, y_0)$$

can only retain a constant sign for small values of h if the derivative $\dfrac{\partial f}{\partial x}$ is zero for $x=x_0$, $y=y_0$.

* See also Goursat, loc. cit.

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In the same way it may be shown that these values must also cause $\frac{\partial f}{\partial y}$ to be zero. It follows that the systems of values which cause $f(x, y)$ to become proper extremes are to be found among the solutions of the two simultaneous equations

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0, \tag{α}$$

conditions which are also necessary for improper extremes.

As only ordinary extremes are considered here, the partial derivatives of the second order of $f(x, y)$ are supposed to be continuous (§ 11) in the neighborhood of the values x_0, y_0 and are not all zero for x_0, y_0, and, furthermore, the derivatives of the third order are supposed to exist. If, then, $x = x_0$ and $y = y_0$ are a solution of the two equations (α), the formula for Taylor's theorem gives us

$$\begin{aligned}\Delta &= f(x_0 + h, y_0 + k) - f(x_0, y_0) \\ &= \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x_0^2} + 2hk \frac{\partial^2 f}{\partial x_0 \partial y_0} + k^2 \frac{\partial^2 f}{\partial y_0^2}\right) \\ &\quad + \frac{1}{3!}\left[\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^3 f(x, y)\right]_{\substack{x = x_0 + \theta h \\ y = y_0 + \theta k}}.\end{aligned} \tag{β}$$

For values of h and k in the neighborhood of zero, it is clear that the trinomial

$$h^2 \frac{\partial^2 f}{\partial x_0^2} + 2hk \frac{\partial^2 f}{\partial x_0 \partial y_0} + k^2 \frac{\partial^2 f}{\partial y_0^2}$$

gives its sign to the right-hand side of (β), and it is evident that the discussion of the sign of this trinomial is going to enjoy a preponderant rôle.

To have an extreme for $x = x_0$, $y = y_0$, it is necessary and sufficient that the difference Δ retain a constant sign when the point $(x_0 + h, y_0 + k)$ remains within the interior of a square sufficiently small which has the point (x_0, y_0) for center. In this case the difference Δ will also retain a constant sign if the point $(x_0 + h, y_0 + k)$ remains within a circle with radius sufficiently small and center (x_0, y_0), and inversely; for we may replace the square by the inscribed circle and reciprocally. Suppose, then, that C is a circle of radius r with the point

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(x_0, y_0) as center. We have all the interior points of this circle by writing $h = \rho \cos\phi$, $k = \rho \sin\phi$ and causing ϕ to vary from 0 to 2π and by causing ρ to vary from $-r$ to $+r$.

Making this substitution in Δ, it becomes

$$\Delta = \frac{\rho^2}{2!}(A\cos^2\phi + 2B\sin\phi\cos\phi + C\sin^2\phi) + \frac{\rho^3}{3!}L,$$

where $A = \frac{\partial^2 f}{\partial x_0^2}$, $B = \frac{\partial^2 f}{\partial x_0 \partial y_0}$, $C = \frac{\partial^2 f}{\partial y_0^2}$, and where L is an expression which retains a finite value in the neighborhood of the point (x_0, y_0).

It is evident that several cases are to be distinguished according to the sign of $B^2 - AC$.

17. *First case.* $B^2 - AC > 0$.

The equation $A\cos^2\phi + 2B\sin\phi\cos\phi + C\sin^2\phi = 0$ admits two real roots in $\tan\phi$, and the left-hand side may be written as the difference of two squares in the form

$$\Delta = \frac{\rho^2}{2}[\alpha(a\cos\phi + b\sin\phi)^2 - \beta(a'\cos\phi + b'\sin\phi)^2] + \frac{\rho^3}{6}L,$$

where $\alpha > 0$, $\beta > 0$, and $(ab' - ba') \neq 0$.

By taking the circle sufficiently small we may neglect the terms of the third and higher degrees in ρ. If next to the angle ϕ a value is given such that $a\cos\phi + b\sin\phi = 0$, it is seen that Δ will be negative; while if we give the angle ϕ a value such that $a'\cos\phi + b'\sin\phi = 0$, then Δ will be positive.

It is therefore impossible to find a number r such that the difference Δ retains a constant sign when the absolute value of ρ is inferior to r, while the angle ϕ is arbitrary. It follows that the function $f(x, y)$ has no extreme for $x = x_0$, $y = y_0$.

18. *Second case.* $B^2 - AC < 0$.

It is evident that A and C must have the same sign.

The trinomial

$$A\cos^2\phi + 2B\cos\phi\sin\phi + C\sin^2\phi \equiv \frac{1}{A}[(A\cos\phi + B\sin\phi)^2 + (AC - B^2)\sin^2\phi]$$

does *not* vanish when ϕ varies from 0 to 2π.

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Let m be the lower limit of the absolute value of the trinomial and let H be the upper limit of the absolute value of the function L in a circle of radius R and center (x_0, y_0).

Let r be a positive number inferior to R and to $\frac{3m}{H}$. Within the circle of radius r the difference Δ will have the same sign as the coefficient of ρ^2, that is to say, of A or C. The function $f(x, y)$ has therefore an extreme for $x = x_0$, $y = y_0$.

19. The above results may be summarized as follows: If at the point x_0, y_0 we have

$$\left(\frac{\partial^2 f}{\partial x_0 \partial y_0}\right)^2 - \frac{\partial^2 f}{\partial x_0^2}\frac{\partial^2 f}{\partial y_0^2} > 0,$$

there is *no* extreme; but if

$$\left(\frac{\partial^2 f}{\partial x_0 \partial y_0}\right)^2 - \frac{\partial^2 f}{\partial x_0^2}\frac{\partial^2 f}{\partial y_0^2} < 0,$$

there is a maximum or minimum according to the sign of the two derivatives $\frac{\partial^2 f}{\partial x_0^2}, \frac{\partial^2 f}{\partial y_0^2}$.

There is a *maximum* if these derivatives are negative, a *minimum* if they are positive, and it is also seen that we have a *proper* maximum or minimum.*

Example. In the theory of least squares it is required to determine x, y so that the expression

$$(A) \qquad u(x, y) = \sum_{k=1}^{k=n} (a_k x + b_k y + c_k)^2$$

may be as small as possible. In other words, determine the values of x and y for which $u(x, y)$ is equal to its lower limit.

Following the methods indicated above we must solve the two equations

$$(B) \qquad \begin{cases} \frac{1}{2}\frac{\partial u}{\partial x} = \sum_{k=1}^{k=n} (a_k x + b_k y + c_k)\, a_k = 0, \\ \frac{1}{2}\frac{\partial u}{\partial y} = \sum_{k=1}^{k=n} (a_k x + b_k y + c_k)\, b_k = 0. \end{cases}$$

It is seen that the determinant of these equations is equal to the sum of the $\frac{1}{2}n(n-1)$ squares $(a_k b_l - a_l b_k)^2 (k, l = 1, 2, \cdots, n;\ k < l)$, and this determinant does *not* vanish if among the binomials $a_k x + b_k y$ there are at least

* See Lagrange, *Misc. Taur.*, Vol. I. 1759.

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two which do not differ from each other by a constant factor. Under this assumption the two equations (B) have one and only one system of solutions x_0, y_0.

That u does in fact reach its lower limit for this pair of values is seen if we write in (A) $x = x_0 + \zeta$, $y = y_0 + \eta$, and expand. We then have

$$u(x_0 + \zeta, y_0 + \eta) - u(x_0, y_0) = \sum_{k=1}^{k=n} (a_k\zeta + b_k\eta)^2,$$

which difference is a *positive* quantity for every system of values except $\zeta = 0$, $\eta = 0$.

PROBLEMS

1. Find a point P of a plane such that the sum $PA + PB + PC$ of its distances to three fixed points of the plane is a minimum. In particular consider the case when $BAC > 120°$, and show that here the point A gives the minimum. (Cavalieri, *Exercitationes Geometricae*, pp. 504–510. 1647.)

2. In a plane triangle all of the angles have been measured with the same precision and found to have values α, β, γ. On account of the unavoidable error in observation, the sum $\alpha + \beta + \gamma$ does not equal 180°. Let the difference $180 - (\alpha + \beta + \gamma)$ be equal to δ, where δ is expressed in circular measure. What values u, v, w (in circular measure) must be added to the three results of measurement if we wish

(1) that $\alpha + \beta + \gamma + u + v + w = 180$, and

(2) that $u^2 + v^2 + w^2$ be as small as possible?

Answer. $u = \frac{1}{3}\delta = v = w$.

Introduction to the Ambiguous Case $B^2 - AC = 0$

20. We shall first note the difficulties that attend this special case, and with Goursat* we shall illustrate these difficulties by means of geometric considerations; we shall then call attention to erroneous deductions which have been made, and later a method will be given, due to Scheeffer, of determining the extremes for this case, when they exist.

Let S be the surface represented by the equation $z = f(x, y)$. If the function $f(x, y)$ has an extreme at the point x_0, y_0, and if the function and its derivatives are continuous, we must have

$$\frac{\partial f}{\partial x_0} = 0, \qquad \frac{\partial f}{\partial y_0} = 0,$$

* Goursat, p. 112.

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which shows that the tangential plane to the surface S at the point P_0 (with coördinates x_0, y_0, z_0) must be parallel to the xy-plane. In order that this point shall correspond to an extreme, it is necessary that in the neighborhood of the point P_0 the surface S be entirely on one side of the tangential plane. We are thus led to the study of a surface with regard to a tangential plane in the neighborhood of the point of contact.

Suppose that the origin has been transposed to the point of contact. The tangential plane being taken as the xy-plane, the equation of the surface is of the form

$$z = ax^2 + 2\,bxy + cy^2 + \alpha x^3 + 3\,\beta x^2 y + 3\,\gamma xy^2 + \delta y^3, \qquad (i)$$

where a, b, c are constants and α, β, γ, δ are functions of x, y which remain finite when x and y tend towards zero. To determine whether the surface S is situated entirely on one side of the xy-plane in the neighborhood of the origin, it is sufficient to study the intersection of this surface by the xy-plane. This intersection is a curve C represented by the equation

$$f(x, y) = ax^2 + 2\,bxy + cy^2 + \alpha x^3 + \cdots = 0, \qquad (ii)$$

and presents a double point at the origin.

21. If $b^2 - ac$ is positive, the equation

$$ax^2 + 2\,bxy + cy^2 \equiv \frac{1}{a}\left[(ax + by)^2 - (b^2 - ac)\,y^2\right] = 0$$

represents two real and distinct straight lines which pass through the origin. Suppose that we take these two lines for the axes of coördinates, and note that this is brought about by a linear change of the variables.

The equation (ii) then has the form

$$xy + R(x, y) = 0. \qquad (iii)$$

If in this equation we write $y = ux$, we have

$$u = -\frac{R(x, ux)}{x^2}, \qquad (iv)$$

where it is evident that $R(x, ux)$ is divisible by x^3.

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It follows from § 140 (see also Goursat, § 34) that equation (*iv*) has one and only one root, say $u = \zeta(x)$, which tends towards zero with x. Hence through the origin there passes one branch of the curve C represented by an equation $y = x\zeta(x)$, which is tangent at the origin to the axis Ox. If we interchange the rôle of the two variables x and y, it is seen that there also passes through the origin a second branch of the curve C which is tangent to the axis Oy. The point O is *a double-point with distinct tangents.*

If, then, $b^2 - ac > 0$, the intersection of the surface S by the tangential plane presents two distinct branches of curve C_1 and C_2 which pass through the origin, and the tangents to these two branches of curve at the origin are represented through the equation

$$ax^2 + 2bxy + cy^2 = 0.$$

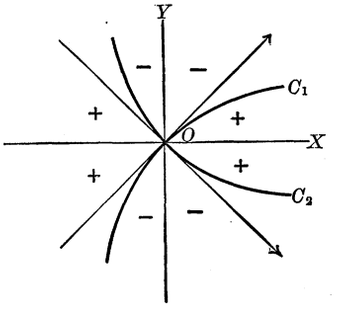

FIG. 7

If we give to each region of the plane in the neighborhood of the origin the sign of the first term in (*iii*), as seen in the figure, it is clear that if a point moves along either of the curves C_1 or C_2, the left-hand side of (*iii*), and consequently also of (*ii*), changes sign as the point passes through the origin. It follows that $f(x, y)$ does *not* have an extreme (cf. § 17) at the origin.

22. If $b^2 - ac < 0$, the origin is a *double isolated* point; for within the interior of a circle with sufficiently small radius described about the origin as center, the right-hand side of (*ii*) only vanishes at the origin itself. To show this write $x = \rho \cos \phi$, $y = r \sin \phi$, where x and y are the coördinates of a point in the neighborhood of the origin.

Equation (*ii*) becomes

$$f(x, y) = \rho^2(a \cos^2\phi + 2b \sin \phi \cos \phi + c \sin^2\phi + \rho L)$$

where L is a function of ρ and ϕ which remains finite when ρ tends towards zero. Let H be the upper limit of $|L|$ when ρ is less than a positive number r.

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When ϕ varies from 0 to 2π, the trinomial

$$a\cos^2\phi + 2b\sin\phi\cos\phi + c\sin^2\phi$$

retains a constant sign. Let m be the minimum of its absolute value. It is clear that the coefficient of ρ^2 does not vanish for any point on the interior of a circle C with radius less than r and $\frac{m}{H}$, having the origin as center. Consequently the equation $f(x, y) = 0$ admits of no other solution than $x = 0, y = 0$ (i.e., $\rho = 0$) within the circle.

It follows that $f(x, y)$ retains a constant sign when the point x, y moves within the interior of this circle. Hence, also, all the points excepting the origin of the surface S which may be projected upon the circle C are situated upon the same side of the xy-plane. The function $f(x, y)$, therefore, presents an extreme at the origin (cf. § 18).

23. When $b^2 - ac = 0$, the two tangents at the double point coincide, and there are, in general, two branches of curve tangent to the same straight line, which form a cusp.

The complete study of this theory will be found to require a somewhat delicate discussion.

For example, $y^2 = x^3$ presents at the origin a cusp of the *first kind*; that is, one which has the two branches of curve tangent to the Ox-axis lying the one above and the other below this tangent.

The curve $y^2 - 2x^2y + x^4 - x^5 = 0$ presents a cusp of the *second kind*; the two branches of curve are tangent to the x-axis and situated on the same side of it. The equation gives us, in fact, $y = x^2 \pm x^{\frac{5}{2}}$. The two values of y have the same sign in the neighborhood of the origin and are only real when x is positive.

The curve $x^4 + x^2y^2 - 6x^2y + y^2 = 0$ presents two branches of curve which offer nothing peculiar, both being tangent at the origin to the x-axis. We have from this equation

$$y = \frac{3x^2 \pm x^2\sqrt{8 - x^2}}{1 + x^2},$$

from which it is seen that the two branches obtained when we take successively the two signs before the radical have *no* singularity at the origin.

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It may also happen that the curve is composed of two coincident branches, as is the case of the curve represented by the equation

$$f(x, y) = y^2 - 2x^2y + x^4 = 0\,; \text{ that is, } (y - x^2)^2 = 0.$$

It is evident that here the left-hand side passes through zero without changing sign.

It may also occur that the point x_0, y_0 is a double isolated point, as is presented through the curve

$$y^2 + x^4 + y^4 = 0$$

at the origin.

From the above it is seen that if the origin is a double isolated point, or if the intersection of the surface with the tangent plane is composed of two coincident branches, the function $f(x, y)$ will be an extreme (in the latter case just given an improper extreme); but if the intersection is composed of two distinct branches which pass through the origin, there will, in general, be no extreme, for the surface again cuts its tangential plane.

24. Take, for example,* the surface

$$z = (y - x^2)(y - 2x^2),$$

which cuts its tangential plane along two parabolas of which the one is interior to the other. That the surface may not cross its tangential plane, it is necessary that if we cut this surface by any cylinder having its elements parallel to Oz and passing through Oz, the curve of intersection shall lie on one side of the xy-plane.

Let $y = \phi(x)$ be the trace of such a cylinder upon the xy-plane, the function $\phi(x)$ being zero for $x = 0$. If $f(0, 0)$ is to be a minimum, the function $f(x, \phi(x)) = F(x)$, say, ought to be a minimum for $x = 0$, whatever the function $\phi(x)$.

To simplify the calculation, suppose that we have chosen the axes of coördinates so that the equation of the surface is of the form

$$z = Ay^2 + \phi_3(x, y) + \cdots,$$

where A is a positive quantity.

* This is a generalization due to Goursat (p. 115) of the classic example of Peano (loc. cit., Nos. 133-136).

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With this system of axes we have for the origin

$$\frac{\partial f}{\partial x_0}=0,\quad \frac{\partial f}{\partial y_0}=0,\quad \frac{\partial^2 f}{\partial x_0^2}=0,\quad \frac{\partial^2 f}{\partial x_0 \partial y_0}=0,\quad \frac{\partial^2 f}{\partial y_0^2}>0.$$

The derivatives of $F(x)$ are

$$F'(x)=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\phi'(x),$$

$$F''(x)=\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x \partial y}\phi'(x)+\frac{\partial^2 f}{\partial y^2}\phi'^2(x)+\frac{\partial f}{\partial y}\phi''(x), \text{ etc.}$$

For $x=0=y$ these formulas become

$$F'(0)=0,\quad F''(0)=\frac{\partial^2 f}{\partial y_0^2}[\phi'(0)]^2.$$

If $\phi'(0)\neq 0$, the function $F(x)$ evidently has a minimum for $x=0$; but if $\phi'(0)=0$, it is seen that

$$F'(0)=0,\quad F''(0)=0,\quad F'''(0)=\frac{\partial^3 f}{\partial x_0^3}$$

and

$$F^{(iv)}(0)=\frac{\partial^4 f}{\partial x_0^4}+6\frac{\partial^3 f}{\partial x_0^2 \partial y_0}\phi''(0)+3\frac{\partial^2 f}{\partial y_0^2}[\phi''(0)]^2.$$

Hence, in order that $F(x)$ be a minimum, it is necessary that $\frac{\partial^3 f}{\partial x_0^3}$ be zero, while

$$\frac{\partial^4 f}{\partial x_0^4}+6\frac{\partial^3 f}{\partial x_0^2 \partial y_0}\phi''(0)+3\frac{\partial^2 f}{\partial y_0^2}[\phi''(0)]^2$$

must be positive, whatever the value of $\phi''(0)$.

These conditions are *not* satisfied for the surface

$$z=y^2-3x^2y+2x^4$$

considered above, while they are satisfied for the function

$$z=y^2+x^4.$$

It is thus seen that in the ambiguous case, where $B^2-AC=0$, the derivation of the necessary and sufficient conditions for the extremes of functions of only two variables is going to be accompanied by difficulties. It is also evident that in the case of three or more variables these difficulties will be correspondingly augmented.

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II. INCORRECTNESS OF DEDUCTIONS MADE BY EARLIER AND MANY MODERN WRITERS

25. One of the greatest mathematicians of all times, Lagrange (*Théorie des Fonctions*, p. 290), writes:

> If all the terms of the first and second dimensions [see formula (*ii*) of § 16] vanish, it is necessary for the existence of a maximum or minimum that all the terms of the third dimension in $h_1, h_2, \cdots$ shall disappear and that the quantity composed of terms where $h_1, h_2, \cdots$ (cf. § 51) form four dimensions shall be always positive for the minimum and always negative for the maximum when $h_1, h_2, \cdots$ have any values whatever.

Following Lagrange, all writers on this subject made the same incorrect deductions until Peano, in the remarks to Nos. 133–136 found in the Appendix to his *Calcolo*, wrote: "The proofs for the criteria by which the maxima and minima of functions of several variables are to be recognized, and which are given in most books, depend upon the theorem that in the Taylor development for functions of several variables the ratio of the remainder after an arbitrary term to this term has a limit zero when the increments of the variables approach zero. This theorem is in general false when the term in question is *not* a definite form with respect to the increments of the variables, and when it is a definite form, the theorem needs proof."

These fallacious conclusions are found, for example, in Bertrand (*Calcul Différentiel*, p. 504), and also in Serret (*Calc.*, p. 219), who writes:

> The maxima or minima exist if for the values $h_1, h_2, \cdots$ which cause d^2f and d^3f to vanish the derivative d^4f has invariably the minus or plus sign.

Here d^2f, $d^3f, \cdots$ denote the homogeneous integral forms of the second, third, $\cdots$ degrees in $h_1, h_2, \cdots$, when the function f is expanded by Taylor's theorem (cf. § 51).

Todhunter (pp. 227–229 of the 1864 and 1881 editions of his Calculus), for the semi-definite case where $B^2 - AC = 0$, writes the Taylor expansion for a function of two variables in the form (see (*ii*) of § 16)

$$\Delta = \frac{k^2}{A}\left(A\frac{h}{k} + B\right)^2 + (\quad)_3 + R_4,$$

where R_4 is the remainder term.

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The condition which it appears that he considered as sufficient for an extreme is that A and R_4 must have the same sign, and if the terms of the second dimension are zero for the position or positions in question, then also the terms of the third dimension must be zero.

That this is *not* true is seen at once by observing Peano's classic example $f(x, y) = (y - p^2x^2)(y - q^2x^2)$,

where the conditions just mentioned exist, although there is no extreme at the origin, as already seen in § 24.

Professor Pierpont (*Bull. of the Am. Math. Soc.*, Vol. IV, p. 536) says, "Our English and American authors seem to be ignorant of these facts."

Write Peano's example in the form

$$f(x, y) = ay^2 + 2bx^2y + cx^4.$$

It is seen that the function $f(x, y)$ is *positive* in the neighborhood of the origin upon every straight line through it; however, upon the parabola $y = mx^2$ the function in the neighborhood of the origin is positive, zero, or negative according as $am^2 + 2bm + c$ is positive, zero, or negative.

We may further illustrate this as follows: Let the equations

$$\Phi(x, y) = y - \phi(x) = 0,$$
$$\Psi(x, y) = y - \psi(x) = 0$$

denote two curves through the origin. The function

$$f(x, y) = \Phi(x, y)\Psi(x, y)$$

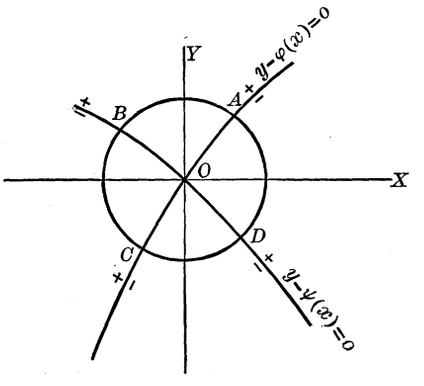

FIG. 8

will have positive values for values of x, y on the arc BA of a circle with origin at the center and radius sufficiently small and negative values on the arc AD. Hence the function $f(x, y)$ has minimum values on all straight lines through the origin that cut the arc BA and maximum values on the lines through the origin that cut the arc AD.

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If, further, the two curves $\Phi(x, y) = 0, \Psi(x, y) = 0$ have a common tangent at the origin with their curvatures lying in the same direction, it is seen that all possible straight lines through the origin are such that the coördinates of any points on them cause $f(x, y)$ to have positive values. This is true, for example, of the function already considered,

$$f(x, y) = (y - p^2x^2)(y - q^2x^2).$$

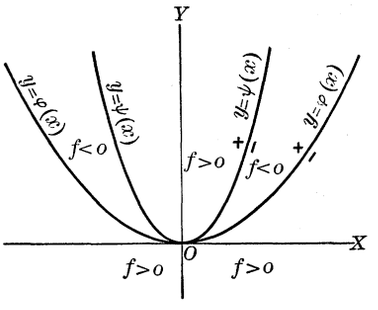

FIG. 9

In the spaces above and below both curves we have $f(x, y) > 0$, while this function is negative for the spaces between the two curves; so that there is a minimum upon every straight line through the origin, although there is a maxiumm* of $f(x, y)$ for all points on the curve $y = \frac{p^2 + q^2}{2} x^2$.

III. DIFFERENT ATTEMPTS TO IMPROVE THE THEORY

26. The existence of an extreme of the function $f(x, y)$ at the origin, for example, a minimum, depends upon the condition that there exists an upper limit g such that the function $f(x, y)$ for all values of x, y which satisfy the condition

$$0 < \sqrt{x^2 + y^2} < g$$

is positive; or, geometrically speaking (§ 16), this condition implies that there exists a circle with center (0, 0) within which the function is everywhere positive with the exception of the position (0, 0) itself.

Instead of considering the values of such a function for the coördinates of points on straight lines through the origin, which lines may be written in the form

$$x = \alpha k, \qquad y = \beta k,$$

* Note in this connection Scheeffer, *Math. Ann.*, Vol. XXVI, p. 197; and Vol. XXXV, p. 545.

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α, β being arbitrary constants, it would be natural to raise the question whether we could not determine the sufficient conditions for such extremes by studying the more general curves expressed through the algebraic equations

$$\left.\begin{aligned} x(k) &= \alpha_1 k + \alpha_2 k^2 + \cdots + \alpha_m k^m \\ y(k) &= \beta_1 k + \beta_2 k^2 + \cdots + \beta_n k^n \end{aligned}\right\}, \qquad (i)$$

and make the requirement that the function $f(x(k), y(k))$ shall have an extreme for $k = 0$, whatever values there may be assigned to the positive integers m and n and to the $m+n$ quantities α_1, $\alpha_2, \cdots, \alpha_m$; $\beta_1, \beta_2, \cdots, \beta_n$, it being of course assumed that all the quantities α and β are not simultaneously zero.

It may, however, be shown that such sufficient conditions *cannot* be derived in the manner indicated. For if we write

$$\Phi(x, y) = y - \sin^2 x,$$
$$\Psi(x, y) = y - \sin^2 x - e^{-\frac{1}{x^2}},$$

we have two curves defined through the equations $\Phi(x, y) = 0$ and $\Psi(x, y) = 0$ which have at the origin the x-axis as a common tangent and a contact of an indefinitely high order.

There is consequently no curve of the form (i) which may be laid between these two curves; for clearly any such curve must have with either of these curves a contact of indefinitely high order which is impossible for an algebraic curve.

On the other hand, the function $f(x, y) = \Phi(x, y)\Psi(x, y)$ is positive in the whole plane excepting that part of the plane that is situated between the two transcendental curves, in which it is negative. Hence at the origin there is neither a maximum nor a minimum for the function $f(x, y)$, although for this function upon every curve (i) there enters a minimum.

We may therefore desist from further requirements in this direction, and we shall next call attention to two methods, the one due to Scheeffer and the other to Von Dantscher, which are general in character when the discussion has to do with two variables and which lead to criteria which are of use in practice.

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27. Scheeffer's method. We have seen that functions of one variable which have ordinary extremes can be expressed through the Taylor development in the form

$$f(x) = \frac{f^{(n)}(\theta x)}{n!} x^n \quad (0 < \theta < 1), \tag{α}$$

when $f(x)$ and the derivatives $f'(x), \cdots, f^{(n-1)}(x)$ are zero for $x = 0$ while $f^{(n)}(x) \neq 0$ for $x = 0$. For such functions the change in value in the neighborhood of the position $x = 0$ on either side is faster than that of a given quantity ax^n; that is, positive quantities a, n, and δ may be so chosen that for all values of x within the interval $-\delta$ to $+\delta$ the absolute value of $f(x)$ is greater than the absolute value of ax^n, excepting the value $x = 0$. For since $f^{(n)}(0) \neq 0$, we may so determine δ that for values of x such that $-\delta \leqq x \leqq \delta$ the function $f^{(n)}(x)$ is different from zero. If, then, we choose the quantity a smaller than the absolute value of $\frac{f^{(n)}(x)}{n!}$ in the interval $-\delta$ to $+\delta$, then (see formula (α)) within this interval the condition $|f(x)| > |ax^n|$ is satisfied. Reciprocally, if the last condition exists, the n first derivatives of $f(x)$ cannot all vanish for $x = 0$. For in the latter case we would have

$$f(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1},$$

and from this it follows that

$$\lim_{x=0} \frac{f(x)}{ax^n} = 0,$$

which contradicts the assumption that $|f(x)| > |ax^n|$.

There are functions, however, for example $e^{-\frac{1}{x^2}}$ (cf. Pierpont, loc. cit., Vol. I, p. 205), for which such quantities n, a, δ, do not exist. In fact, the absolute value of $e^{-\frac{1}{x^2}}$ is in the immediate neighborhood of $x = 0$ smaller than any arbitrary power ax^n.

We may note that the characteristic property of the above requirement consists in the fact that the behavior of the function in the neighborhood of the origin must be marked with a certain degree of *distinctness.*

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The following consideration leads to the generalization of the above condition for functions of two variables: It is clear that a function $f(x, y)$ which vanishes at the origin, if it is continuous, has upon the circumference of every circle which is described about the origin as center with an arbitrary radius r a greatest and a least value, provided the function does *not* reduce to a function of one variable $r = \sqrt{x^2 + y^2}$. The signs of these greatest and least values, which we denote by $f_1(r)$ and $f_2(r)$ respectively, offer for sufficiently small radii r a criterion regarding the appearance or nonappearance of an extreme at the origin.* For if the two quantities $f_1(r)$ and $f_2(r)$ are positive, there will be a minimum of $f(x, y)$ at the origin, while if they are both negative, a maximum exists at the origin. The *degree* of *distinctness* which marks the behavior of the function at the origin is characterized through the existence of a power ar^n with the property that for every value of r within a certain limit g both $f_1(r)$ and $f_2(r)$ are in absolute value greater than ar^n.

If this requirement is *not* satisfied we cannot count upon deriving sure characteristics of extremes through the expansion in series. For in this case the value with which the function $f(x, y)$ in the neighborhood of the position (0, 0) either approaches the value zero from the one side, or having passed through zero differs from it on the other side, is so little that this value cannot be expressed through a power ever so high of r. The development in series cannot, therefore, serve to determine whether the value is a little on the one side or on the other side of zero.

As examples of this kind are the function

$$y^2 + e^{-\frac{p^2}{x^2}},$$

which has a minimum value at the origin, and the function

$$y^2 - e^{-\frac{p^2}{x^2}} = (y + e^{-\frac{p^2}{2x^2}})(y - e^{-\frac{p^2}{2x^2}}),$$

* The behavior of the function $f(x, y)$ at any point x_0, y_0 other than the origin may be made by the substitution $x = x_0 + h$, $y = y_0 + k$, to depend upon the behavior of the function $f(x_0 + h, y_0 + k) = F(h, k)$ for the values $h = 0$, $k = 0$.

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which has neither a maximum nor minimum at the origin. The first function approaches the value zero from the positive direction up to the value $e^{-\frac{p^2}{r^2}}$ (for $y=0$) while the latter approaches the value zero from the negative direction by the same amount.

To this class of functions belong also those functions whose initial terms constitute a semi-definite form and which contain as a factor an even power of a series $P(x, y)$ the terms of which vanish for real pairs of values x, y in every region arbitrarily small where $0<|x|<\delta$, $0<|y|<\delta$ (see §§ 36 and 41). Belonging also to this category of functions are the functions which reach the value zero but do not pass through it for every region arbitrarily small where $0<|x|<\delta$, $0<|y|<\delta$.

If on the other hand there exists a power ar^n whose value, so long as we remain within a certain limit g, is always smaller than the absolute values of $f_1(r)$ and $f_2(r)$, then the question whether at the origin an extreme of the function exists may always be answered by a development in series and by a finite number of observations. How this is accomplished is found in the next chapter.

28. The method of Von Dantscher. We have seen that by considering the extremes on every line through (0, 0) we are not able to form any conclusions regarding the extremes of the function $f(x, y)$ at this point. Von Dantscher's method consists in establishing criteria not only for the extremes on such lines but also for all points in the plane in the neighborhood of the points on these lines and also in the neighborhood and on both sides of the point (0, 0). Although Von Dantscher himself finds that there is "no need of an extension or improvement of the Scheeffer method," I shall give later the method of Von Dantscher, as it is of interest in itself and, besides, it is well to compare the two theories (see §§ 42, 44).

29. The Stolzian theorems.* We shall at first assume that the function $f(x, y)$ is continuous with respect to both variables in every point (x, y) of a rectangle that includes the point (0, 0), the

* Stolz, p. 213.

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sides of the rectangle being parallel to the coördinate axes. We shall state and then prove the following theorems:

THEOREM I. *A necessary condition that $f(0, 0)$ be a proper extreme of $f(x, y)$ is offered through the existence of an interval $-\delta \cdots +\delta$, within which $x(\neq 0)$ lies, and such that the upper limit of $f(x, y)$, when x takes a constant value, the variable y being confined to the interval $+x \cdots -x$, is had through the value $y = \phi_2(x)$, and the lower limit through $y = \phi_1(x)$. This necessary condition in question for a proper maximum is that $f(x, \phi_2(x))$ be invariably less than $f(0, 0)$, and for a proper minimum we must have invariably $f(x, \phi_1(x))$ greater than $f(0, 0)$.*

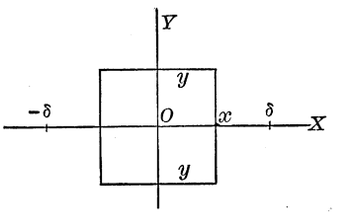

FIG. 10

In the first case the upper and lower limits of $f(x, y)$ are both less than $f(0, 0)$ and in the second case they are both greater.

Note that $\lim_{x=0} \phi_2(x) = 0$, since $|\phi_2(x)| \leqq |x|$. The same is true of $\phi_1(x)$.

The same conditions must be true with regard to the upper and lower limits of $f(x, y)$ with constant y such that $|y| < \delta$, the variable x being limited to the interval $-y \cdots +y$, which limits are reached through the values $\psi_2(y)$ and $\psi_1(y)$ respectively.

THEOREM II. *The fulfillment of all the conditions made above is sufficient that $f(0, 0)$ be a proper extreme of $f(x, y)$. Accordingly $f(0, 0)$ is a proper maximum if there exists a positive quantity δ such that we have simultaneously*

[1] *for* $0 < |x| < \delta$, $f(x, \phi_2(x)) < f(0, 0)$,

and

[2] *for* $0 < |y| < \delta$, $f(\psi_2(y), y) < f(0, 0)$

with corresponding conditions for a proper minimum.

To prove the two theorems just stated we remark first that on account of the continuity of $f(x, y)$ with respect to y the function

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$f(x, y)$ with constant x and with the assumption that y takes all values of the interval $-x \cdots +x$ has for all these values a finite upper limit and a finite lower limit, and further that $f(x, y)$ reaches these limits for values $y = \phi_2(x)$ and $y = \phi_1(x)$ (see § 8).

Hence for values of y such that

[3] $\quad |y| \leqq |x|$ it is clear that $f(x, y) \leqq f(x, \phi_2(x))$.

Furthermore, in virtue of the definition of a proper maximum of $f(x, y)$ there must be a positive quantity δ such that if only $|x|$ and $|y|$ are smaller than δ we must have

$$[4] \qquad f(x, y) - f(0, 0) < 0.$$

It follows, if $|x| < \delta$ and $x \neq 0$ and if we substitute $y = \phi_2(x)$ in [4], that

$$f(x, \phi_2(x)) - f(0, 0) < 0,$$

which is in fact the inequality [1].

Reciprocally from [1] and [3] are obtained the inequalities

$$0 < |x| < \delta \quad \text{and} \quad f(x, y) - f(0, 0) < 0,$$

where

$$|y| \leqq |x| < \delta.$$

If the relation [4] is to be true for all systems of values (x, y) where $|x|$ and $|y|$ are smaller than δ (excepting $x = 0$ and $y = 0$), then in addition to [1] we must have the corresponding pair of inequalities [2], which may be derived without trouble.

We have corresponding conditions for *improper* extremes:

THEOREM III. *In order that $f(0, 0)$ be an improper maximum of $f(x, y)$ it is necessary and sufficient that there exist a positive quantity δ such that for any x with absolute value less than δ the value $f(x, \phi_2(x))$ is* **not** *greater than $f(0, 0)$ and for any y with absolute value less than δ the value $f(\psi_2(y), y)$ is* **not** *greater than $f(0, 0)$; while at the same time corresponding to every positive quantity δ' which is less than δ there is at least one value of x or y whose absolute value is less than δ' and for which either $f(x, \phi_2(x))$ or $f(\psi_2(y), y)$ is equal to $f(0, 0)$. The conditions for an improper minimum follow at once.*

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THEOREM IV. That $f(0, 0)$ may *not* be a minimum (proper or improper) of $f(x, y)$ it is necessary and sufficient that to every positive quantity δ there either exists a quantity x', with absolute value less than δ, such that

$$[5] \qquad f(x', \phi_1(x')) < f(0, 0),$$

or that there exist a quantity y', with absolute value less than δ, such that

$$[6] \qquad f(\psi_1(y'), y') < f(0, 0);$$

and that $f(0, 0)$ may *not* be a maximum (proper or improper) of $f(x, y)$ it is necessary and sufficient that corresponding to every positive quantity δ there may be found either a quantity x'', with absolute value less than δ, such that

$$[7] \qquad f(x'', \phi_2(x'')) > f(0, 0),$$

or a quantity y'', with absolute value less than δ, such that

$$[8] \qquad f(\psi_2(y''), y'') > f(0, 0).$$

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CHAPTER IV

THE SCHEEFFER THEORY

I. GENERAL CRITERIA FOR A GREATEST AND A LEAST VALUE OF A FUNCTION OF TWO VARIABLES; IN PARTICULAR THE EXTRAORDINARY EXTREMES

30. The theorems of Stolz which were developed in the preceding article are closely related to those of Scheeffer, which are of more practical value since the computations required have to do mostly with a few of the initial terms of the expansion of $f(x, y)-f(0, 0)$ in ascending positive integral powers of x and y. We shall assume that the function $f(x, y)$ is such that it may be expanded by the Taylor-Lagrange theorem in the form

$$\begin{aligned} &f(x+h, y+k) \\ &\quad = f(x, y)+[hf'_x(x+\theta h, y+\theta k)+kf'_y(x+\theta h, y+\theta k)] \\ &\quad = f(x, y)+hf'_x(x, y)+kf'_y(x, y)+\tfrac{1}{2}[h^2f''_{xx}(x+\theta h, y+\theta k) \\ &\qquad +2hkf''_{xy}(x+\theta h, y+\theta k)+k^2f''_{yy}(x+\theta h, y+\theta k)], \text{ etc.,} \end{aligned}$$

where $0<\theta<1$.

If we write $x=0$, $y=0$ and then put $h=x$, $k=y$, it is seen that

$$[1] \qquad f(x, y)-f(0, 0)=G_n(x, y)+R_{n+1}(x, y),$$

where $G_n(x, y)$ denotes the collectivity of terms of the n first dimensions and $R_{n+1}(x, y)$ is the remainder term (Lagrange, *Théorie des Fonctions*, Vol. I, p. 40).

The Scheeffer theorem. *If an index n and positive quantities a and δ can be determined to satisfy the two postulates (1) that for all values of x such that $0<|x|<\delta$ the upper and lower limits of $|G_n(x, y)| \gtrless a|x|^n$, with constant values of x and with y limited to the interval $-x \cdots +x$, and (2) that for all values of y such*

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that $0<|y|<\delta$ *the upper and lower limits of* $|G_n(x, y)| \equiv a|y|^n$, *where* y *has constant values and where* x *lies within the interval* $-y \cdots +y$, *then the two functions* $f(x, y)$ *and* $G_n(x, y)$ *have simultaneously on the position* $(0, 0)$ *either a proper maximum or a proper minimum.*

For, let the lower and upper limits of $G_n(x, y)$, with constant x and with $|y| \equiv |x|$, be $G_n(x, \Phi_1(x))$ and $G_n(x, \Phi_2(x))$ (see § 29); and with constant y and with $|x| \equiv |y|$ let the upper and lower limits of $G_n(x, y)$ be $G_n(\Psi_2(y), y)$ and $G_n(\Psi_1(y), y)$. Since $R_{n+1}(x, y)$ is a homogeneous integral function of the $(n+1)$th dimension in x, y and consists of $n+2$ terms, we note that corresponding to any positive quantity ϵ' we may always find another positive quantity δ' such that if

$$|y| \equiv |x| \text{ and } 0<|x|<\delta', \text{ then } |R_{n+1}(x, y)|<(n+2)\epsilon'|x||x|^n;$$

and also such that if

$$|x| \equiv |y| \text{ and } 0<|y|<\delta', \text{ then } |R_{n+1}(x, y)|<(n+2)\epsilon'|y||y|^n.$$

Hence writing $(n+2)\epsilon'|x| = \epsilon$ and $(n+2)\epsilon'|y| = \epsilon$, and denoting the corresponding value of δ' by δ, it is seen that there is always an interval $-\delta \cdots +\delta$ such that if

[2] $0<|x|<\delta$ and $|y| \equiv |x|$, then $|R_{n+1}(x, y)|<\epsilon|x|^n$;

and if

[3] $0<|y|<\delta$ and $|x| \equiv |y|$, then $|R_{n+1}(x, y)|<\epsilon|y|^n$.

It follows then from [1] and [2] that for values of x, y such that $|x|<\delta$ and $|y| \equiv |x|$ we have

$$[4] \qquad G_n(x, \Phi_1(x)) - \epsilon|x|^n < f(x, y) - f(0, 0) < G_n(x, \Phi_2(x)) + \epsilon|x|^n;$$

and from [1] and [3] that for values of x, y such that $0<|y|<\delta$ and $|x| \equiv |y|$ we have

$$[5] \qquad G_n(\Psi_1(y), y) - \epsilon|y|^n < f(x, y) - f(0, 0) < G_n(\Psi_2(y), y) + \epsilon|y|^n.$$

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If next we assume that $G_n(0, 0)$ is a proper extreme of $G_n(x, y)$ and that the two postulates of the theorem have been satisfied, then if $G_n(0, 0)$ is a minimum it is evident for small values of x and y that $G_n(x, \Phi_1(x))$ and $G_n(\Psi_1(y), y)$ are positive quantities, and from the postulates it follows that for values

$$0 < |x| < \delta \text{ and } |y| \leqq |x| \text{ we have } G_n(x, \Phi_1(x)) \geqq a|x|^n$$

and for values

$$0 < |y| < \delta \text{ and } |x| \leqq |y| \text{ we have } G_n(\Psi_1(y), y) \geqq a|y|^n.$$

Accordingly it follows from [4] for values

[6] $0 < |x| < \delta$ and $|y| \leqq |x|$ that $(a - \epsilon)|x|^n < f(x, y) - f(0, 0)$;

and from [5] for values

[7] $0 < |y| < \delta$ and $|x| \leqq |y|$ that $(a - \epsilon)|y|^n < f(x, y) - f(0, 0)$.

Since ϵ may be made smaller than a, it follows in both [6] and [7] that $f(x, y) - f(0, 0)$ is positive and consequently that $f(0, 0)$ is a proper minimum of $f(x, y)$ (see Stolz's second theorem, § 29).

If $G_n(0, 0)$ is a proper maximum of $G_n(x, y)$, then with small values of x and y the expressions $G_n(x, \Phi_2(x))$ and $G_n(\Psi_2(y), y)$ must be negative.

Hence, due to the postulates for values

$$0 < |x| < \delta \text{ and } |y| \leqq |x|, \text{ we have } G_n(x, \Phi_2(x)) \leqq -a|x|^n,$$

and for values

$$0 < |y| < \delta \text{ and } |x| \leqq |y|, \text{ we have } G_n(\Psi_2(y), y) \leqq -a|x|^n;$$

and in a similar manner as above it follows that $f(0, 0)$ is a proper maximum of $f(x, y)$.

31. Stolz's* added theorem. *If $G_n(0, 0)$ is* ***not*** *an extreme of $G_n(x, y)$, the following conditions are sufficient to make it impossible that $f(0, 0)$ should be an extreme of $f(x, y)$: if (1) for all positive values of x and y such that $0 < |x| < \delta$ and $0 < |y| < \delta$, or for all negative values within the same limits, at least one of the two upper limits of $G_n(x, y)$ defined above is* ***positive*** *and not less than $a|x|^n$ or*

* Stolz, p. 218.

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$a|y|^n$ respectively, and (2) for all positive values of x and y such that $0<|x|<\delta$ and $0<|y|<\delta$, or for all negative values within the same limits, at least one of the two lower limits of $G_n(x, y)$ defined above is **negative** *and not greater than $-a|x|^n$ or $-a|y|^n$ respectively; that is, if, under the restrictions just made, $G_n(x, \phi_2(x))$ is positive and $G_n(x, \Phi_1(x))$ negative, or if $G_n(\Psi_2(y), y)$ is positive and $G_n(\Psi_1(y), y)$ negative.*

If we limit x, for example, to the interval $0 \cdots \delta$, and if we suppose that the following inequalities $G_n(x, \Phi_2(x)) \geqq a|x|^n$ and $G_n(x, \Phi_1(x)) \leqq -a|x|^n$ exist, it is seen that these two expressions vanish only for $x = 0$.

From [1] and [2] it follows for $y = \Phi_1(x)$ and $y = \Phi_2(x)$ that for values of x within the interval in question

$$f(x, \Phi_1(x)) - f(0, 0) < -(a - \epsilon)|x|^n$$

and

$$f(x, \Phi_2(x)) - f(0, 0) > (a - \epsilon)|x|^n.$$

Since we may take $\epsilon < a$, it is seen that in the two expressions just written, the difference on the left-hand side is in the first case negative and in the second case positive, so that $f(0, 0)$ is *not* an extreme of $f(x, y)$ (see Stolz's fourth theorem, § 29).

32. The analytic proof given in § 30 of the Scheeffer theorem is essentially due to Stolz. Owing to its importance we shall give Scheeffer's statement of this theorem with his geometric deductions (*Math. Ann.*, Vol. XXXV, p. 553).

The Scheeffer theorem otherwise stated. *Let $f(x, y)$ be any function as already defined of x, y which vanishes at the origin* and let its behavior in the neighborhood of this point be sufficiently explicit for the determination regarding the appearance of extremes by means of power series to be possible; in other words, we assume that there exists a power ar^n such that upon every circle described about the origin as center, whose radius r is not smaller than a definite quantity g, the greatest and the least values of the function $f(x, y)$, viz., $f_1(r)$ and $f_2(r)$ for all points of the*

* If $f(0, 0) \neq 0$, we must write $f(x, y) - f(0, 0)$ in the place of $f(x, y)$ in the present discussion.

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circumference of the circle with radius r, are in absolute value greater than ar^m. Then in the Taylor-Lagrange development given above

$$f(x, y) = G_n(x, y) + R_{n+1}(x, y),$$

where $R_{n+1}(x, y)$ consists of all terms beyond those of the nth dimension, the integral rational function $G_n(x, y)$ behaves in the neighborhood of the origin as does the function $f(x, y)$. For, as we shall show, in the ***first*** *place the greatest and the least values of both functions correspond with respect to sign for every small radius r, and from this it follows that there appear simultaneously at the origin extremes for both functions, if such extremes exist; and* ***secondly,*** *if a' is any quantity situated between 0 and a, then upon the circumference of every circle with radius r (within a certain limit g') the greatest and the least values of the function $G_n(x, y)$ are in absolute value greater than $a'r^m$, and from this it follows also that the degree of distinctness that marks the behavior of $G_n(x, y)$ is the same as that of $f(x, y)$.*

It is evident that we may replace x and y in the remainder term $R_{n+1}(x, y)$ by r, where r is the radius of the small circle about the origin within which the point (x, y) is situated; and at the same time we may replace all coefficients by their absolute values. In this way we have for the absolute value of $R_{n+1}(x, y)$ an upper limit Ar^{m+1}. We shall take the radius r smaller than $\frac{a}{A}$ so that $ar^m > Ar^{m+1}$.

Since $f_1(r)$ and $f_2(r)$ are by hypotheses greater in absolute value than ar^m, it follows from the equation

$$G_n(x, y) = f(x, y) - R_{n+1}(x, y)$$

that those values of x, y on the periphery of the circle with radius r which give $f_1(r)$ and $f_2(r)$, cause $G_n(x, y)$ and $f_n(x, y)$ to have the same sign. If $f_1(r)$ and $f_2(r)$ have the same sign, it follows from the above expression that the greatest and least values of $G_n(x, y)$ have this same sign. If the two quantities $f_1(r)$ and $f_2(r)$ have contrary signs, the same is true of $G_n(x, y)$ for those values of x, y which produce $f_1(r)$ and $f_2(r)$; and

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consequently for a greater reason the greatest and least values of $G_n(x, y)$ have contrary signs.

The second part of the theorem follows in the same way if we take the radius r not only smaller than $\frac{a}{A}$ but also so small that $ar^n - Ar^{n+1} > a'r^n$; that is, if we put g' equal to $\frac{a-a'}{A}$ and take r less than g'. It is then evident that the values of x, y which produce $f_1(r)$ and $f_2(r)$ when written in the expression

$$G_n(x, y) = f(x, y) - R_{n+1}(x, y)$$

cause the right-hand side to be in absolute value greater than $a'r^n$ when $f_1(r)$ and $f_2(r)$ have the same sign; and when these two quantities have contrary signs the corresponding values of $G_n(x, y)$ will in absolute value be greater than $a'r^n$, and the same must a fortiori be true of the greatest and the least values of $G_n(x, y)$.

33. If, however, we cannot find an integer n and a quantity a' which satisfy the conditions above, we can make no conclusions regarding the behavior of the function $f(x, y)$ by means of powers series and by using the method indicated. For in this case we shall show by means of simple examples which follow this chapter that in some cases the function $G_n(x, y)$ is invariably positive, while $f(x, y)$ may be also negative; and in some cases $G_n(x, y)$ may be both positive and negative while $f(x, y)$ retains a constant sign (see Ex. 3, p. 61, and Prob. 2, p. 62). But if the conditions of Scheeffer's theorem exist it is seen that the investigation of the function $f(x, y)$ has been reduced to that of the function $G_n(x, y)$; in other words, the investigation has resolved itself into the question: *How can we recognize whether a limit g' and a quantity a' exist such that upon every circle with radius $r < g'$ the greatest and the least values of a given integral function of the nth degree $G_n(x, y)$ are in absolute value greater than $a'r^n$? And how can we eventually fix the signs of these greatest and least values and thereby determine the extremes of the function $G_n(x, y)$?*

These questions we shall now answer.

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II. HOMOGENEOUS FUNCTIONS

34. In the expansion of $f(x, y) - f(0, 0)$ suppose that the first terms that appear form a homogeneous function of the nth degree in x and y which is the function $G_n(x, y)$. With respect to such a function there are three cases to consider, according as this function is a definite form, an indefinite form, or a semi-definite form (see § 13). If we write

$$G_n(x, y) = \sum_{i=0}^{i=n} a_i x^{n-i} y^i,$$

it is seen that $G_n(x, y)$ changes upon every straight line through the origin proportionally to the nth power of r. If then G_1 and G_2 are the greatest and the least values of $G_n(x, y)$ upon the periphery of the unit circle, then $G_1 r^n$ and $G_2 r^n$ are the greatest and least values upon any arbitrary circle r.

The signs of G_1 and G_2 may be obtained directly through decomposing $G_n(x, y)$ into its linear factors, which may be found by solving an equation of the nth degree. For we may write

$$G_n(x, y) = x^n G_n\left(1, \frac{y}{x}\right) = x^n g(u), \quad \text{where} \quad \frac{y}{x} = u$$

and $g(u)$ is an integral function of the nth degree in u. Owing to the fundamental theorem of algebra, $g(u)$ may be decomposed into factors which are linear and quadratic with negative discriminants if we restrict all the coefficients to real quantities; or these factors are all linear if we allow imaginary coefficients, the quadratic factors breaking up into two imaginary linear components. If these factors are multiplied by the respective powers of x, we have the corresponding decomposition of $G_n(x, y)$ into its linear and quadratic factors. At the outset it is clear that if the degree of $G_n(x, y)$ is odd, then G_1 and G_2 must be equal but of opposite sign, since $G_n(x, y)$ changes sign when x, y are changed into $-x$, $-y$. Furthermore, note that $G_n(\omega x, \omega y) = \omega^n G_n(x, y)$, where ω is a positive quantity. It follows that if $G_n(x, y)$ is positive, negative, or zero, then $G_n(\omega x, \omega y)$ is positive, negative, or zero.

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If $G_n(x, y)$ is an indefinite form, there are values x, y which give $G_n(x, y)$ a positive value, and other values x, y which give it a negative value. Let δ be a positive quantity however small. It is seen that by a proper choice of ω we may find values of x, y where $|x| < \delta$ and $|y| < \delta$ such that $G_n(x, y)$ is positive, and other systems of values x, y within the same interval for which $G_n(x, y)$ is negative. Accordingly the value $G_n(0, 0)$ is *not* an extreme of $G_n(x, y)$.*

If, however, n is even, and, *first*, if the linear factors of $G_n(x, y)$ are all imaginary, then $G_n(x, y)$ cannot change sign nor vanish. It is a *definite* form and the quantities G_1 and G_2 have the same sign. If, *secondly*, there are real linear factors, and if at least one enters to an odd degree, then $G_n(x, y)$ takes both signs. $G_n(x, y)$ is then an indefinite form and the sign of G_1 is different from that of G_2. If, *thirdly*, there enter real linear factors, but each only to an even degree, the form $G_n(x, y)$ may vanish but it cannot change sign. It is a semi-definite form, and one of the extremes G_1 and G_2 is zero. In this case by a proper choice of ω above it is seen that $G_n(x, y)$ vanishes for values of x, y other than zero and situated within the interval $|x| < \delta$ and $|y| < \delta$. In this case $G_n(0, 0)$ is an improper extreme of $G_n(x, y)$; and the behavior of $f(x, y)$ at the origin cannot be recognized without further discussion.

In all cases† except the last a positive quantity a' may be so determined that upon every arbitrary circle r the greatest and least values of the function $G_n(x, y)$, viz., $G_1 r^m$ and $G_2 r^m$, are in absolute value greater than $a' r^m$; for we need only take a' smaller than the absolute values of G_1 and G_2. In these cases (again excepting the last) there are found in a sufficiently distinct manner (in the previous precise sense of the word, see § 27) either a maximum or a minimum of the function $G(x, y)$, or there does not exist such an extreme.

The decomposition of $G_n(x, y)$ into its linear factors is not necessary, since we may determine the sign of G_1 and G_2 by

* Cf. Stolz, p. 222.

† The discussion is for the most part due to Scheeffer, loc. cit.

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means of elementary algebraic operations. For we may determine the multiple factors of $G_n(x, y)$ and write this function in the form

$$G_n(x, y) = \psi_m^{\lambda_m} \psi_{m-1}^{\lambda_{m-1}} \cdots \psi_2^{\lambda_2} \psi_1^{\lambda_1},$$

where, in general, ψ_k is an irreducible factor of the kth degree in x and y with *integral* coefficients and λ_k denotes the number of times this factor occurs. Then by Sturm's theorem we may determine for each such function $\psi_k(x, y)$ the number of real factors and by λ_k the number of times such factor is repeated.

The theory just outlined of the integral homogeneous functions offers, owing to the Scheeffer theorem for the general theory of maxima and minima of arbitrary functions, the following theorem:

If in the development of the function $f(x, y)$ in powers of x, y all terms of the first to the $(n-1)$th dimensions are identically zero, while the terms of the nth dimension constitute a form $G_n(x, y)$ homogeneous in x and y, and if, first, $G_n(x, y)$ is an ***indefinite*** *form (which is always the case if n is odd), then on the position $(0, 0)$ there is neither a maximum nor a minimum of the function $f(x, y)$; if, secondly, $G_n(x, y)$ is a* ***definite*** *form, there enters according to the sign of this form an extreme of $f(x, y)$; if, finally, $G_n(x, y)$ is* ***semi-definite****, the behavior of the function $f(x, y)$ cannot be recognized from the behavior of $G_n(x, y)$.*

From this theorem it follows that if $f(0, 0)$ is an extreme of $f(x, y)$, the terms of the first dimension of the expansion by Taylor's formula of $f(x, y) - f(0, 0)$ must be wanting, and consequently we must have

$$f'_x(0, 0) = 0 \quad \text{and} \quad f'_y(0, 0) = 0.$$

If, furthermore,

$$f(x, y) - f(0, 0) = Ax^2 + 2Bxy + Cy^2 + R_3(x, y),$$

then $f(0, 0)$ is *not* or *is* (in fact a proper) extreme of $f(x, y)$ according as $AC - B^2$ is negative or positive. If this discriminant is positive, then $f(0, 0)$ is a maximum or a minimum according as A and C (which necessarily have one and the same sign) are negative or positive (see § 14).

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But if $AC - B^2 = 0$, a criterion regarding an extreme of $f(x, y)$ with the help only of the terms of the second dimension cannot be had. We must then take in addition terms of the third, fourth, $\cdots$ degrees in the above expansion of $f(x, y)$ in order, if possible, to satisfy the postulates of Scheeffer regarding the function $G_n(x, y)$. In this case we may write, if A is different from zero,

$$f(x, y) - f(0, 0) = \frac{1}{A}(Ax + By)^2 + P_3(x, y) + P_4(x, y) + \cdots,$$

where P_3, P_4, $\cdots$ denote the collectivity of the terms respectively of the third, fourth, $\cdots$ dimensions in x, y.

If in this expression we write $x = Bt$, $y = -At$, it is seen that

$$f(Bt, -At) - f(0, 0) = A_3 t^3 + A_4 t^4 + \cdots,$$

and if the constant A_3 is different from zero, it is seen that by giving positive and negative values to t, the above expression may take both positive and negative values, so that there is no extreme of $f(x, y)$ on the position (0, 0).

But even if the first term that appears on the right of the expansion in t is of even degree, we cannot conclude that there is an extreme, as is illustrated by the classic example of Peano (see § 24), viz., $f(x, y) = Ay^2 + 2Bx^2y + Cx^4$.

Further investigation is therefore necessary when the terms of the second degree constitute a semi-definite form, and this case is continued in the following sections.

III. EXTREMES OF THE FUNCTION $G_n(x, y)$, INTEGRAL IN x AND y, WHICH IS NOT HOMOGENEOUS

35. We must next determine whether or not the value $G_n(0, 0)$ is an extreme of $G_n(x, y)$ when this function is not homogeneous in x and y and when the terms of the lowest dimension in $G_n(x, y)$ constitute a semi-definite form. We must again raise the question regarding the existence of an expression $a'r^m$ which for all sufficiently small values of r is to be smaller than the absolute values of the greatest value and of the smallest value of $G_n(x, y)$ upon the periphery of a circle of radius r, where r is sufficiently small.

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In order, then, to acquaint ourselves with the different possibilities which may enter in the behavior of the function $G_n(x, y)$ at the point (0, 0), we take a small circle with radius r and seek upon it the two positions at which the function $G_n(x, y)$ takes its greatest and its least value. Call these values the *extreme values* of $G_n(x, y)$. They are found (see § 15) by solving the three equations

$$\frac{\partial G_n}{\partial x} - \lambda x = 0,$$

$$\frac{\partial G_n}{\partial y} - \lambda y = 0,$$

$$x^2 + y^2 = r^2.$$

By eliminating λ from the first two of these equations we have an equation of the nth degree

$$y\frac{\partial G_n}{\partial x} - x\frac{\partial G_n}{\partial y} = 0, \tag{i}$$

an equation which is satisfied by all values of x and y which offer extreme values of $G_n(x, y)$ upon any arbitrary circle r.

It is known in the theory of algebraic functions that every branch of an algebraic curve of the nth order which contains the origin may be expressed in the neighborhood of the origin through an independent variable (k, say) in the form

$$\left.\begin{aligned} x &= a_1 k + a_2 k^2 + \cdots \\ y &= b_1 k + b_2 k^2 + \cdots \end{aligned}\right\}; \tag{ii}$$

and this expression for the curve may be made in any number of different ways such that in each of the series for x and y the first coefficient which is different from zero (in case there is one) has an exponent which is $\equiv n$. It follows that both those branches which include the origin of the curve (*i*), and whose points of intersection with the circles of small radii offer the extreme values of $G_n(x, y)$ upon these circles, may be expressed in the form (*ii*) through an independent parameter k, so long, at least, as we remain in the immediate vicinity of the origin; that is, so long as very small values are ascribed to k. We shall call these two branches the two *extreme curves* of the function $G_n(x, y)$.

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36. We must next distinguish between the cases (1) when (excepting for isolated values of r) the extreme values of $G_n(x, y)$ are both different from zero and (2) when one of these extremes is zero.

If both extremes are different from zero, then the expression $G_n(x, y)$, if we write for x and y the two series (*ii*) which correspond to an extreme curve, will begin with a term Ak^m, which for small values of k determines both the sign and the order of magnitude of the entire expression. This order is the mth order if we consider k a quantity of the first order, and it is of the $\frac{m}{\mu}$th order if we consider k^μ the first order, where k^μ is the smallest exponent that actually appears in (*ii*). The number μ, as we saw above, can at most be equal to n. We have similar quantities A', m', μ' for the second extreme curve. If the two numbers m and m' are not both even, there can be no maximum nor minimum of $G_n(x, y)$ at the origin, since this function in this case changes sign with k upon an extreme curve. The same is true if m and m' are even numbers while A and A' have opposite signs, for then the function $G_n(x, y)$ shows different signs upon the two extreme curves.

If, finally, m and m' are both even while A and A' have the same sign, then we have a maximum or minimum of $G_n(x, y)$ according as this sign is negative or positive.

In all three cases it is clear that a quantity a' and an upper limit g' of r may be so determined that for $r < g'$ the values of $G_n(x, y)$ upon both extreme curves are everywhere in absolute value greater than $a'r^p$, where p is the greater of the two numbers $\frac{m}{\mu}$ and $\frac{m'}{\mu'}$.

If, however, the value of $G_n(x, y)$ is invariably zero upon one of the extreme curves, there cannot be a maximum or minimum at the origin, nor is there an expression $a'r^p$ of the kind required above. But this can only occur when $G_n(x, y)$ contains a squared factor which when put equal to 0 defines a real double curve that passes through the origin; for otherwise, with the vanishing of $G_n(x, y)$ upon crossing the circumference of any circle with

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radius r, there must be a change of sign in $G_n(x, y)$. The squared factor enters as a factor to the first power in (*ii*), so that points on this curve make $G_n(x, y)$ *identically* zero.

In the sequel we shall assume that such factors have been divided out of $G_n(x, y)$, so that the case in question does *not* enter.

Under this assumption, which must be tested in every individual case, there exists, in virtue of the considerations already laid down, always a smallest number p associated with which a constant a' and an upper limit g' of the radius r may be so determined that upon every circle of radius $r < g'$ the two extreme values of $G_n(x, y)$ are in absolute value greater than $a'r^p$; and, in fact, this number p (if the order of r is taken as unity) expresses the degree of the magnitude of the function $G_n(x, y)$ upon that one of the two extreme curves upon which this order is the highest. If p is at most equal to n, then $a'r^p$ for small values of r is *not* smaller than $a'r^n$, and the two extreme values of $G_n(x, y)$ are therefore certainly greater in absolute value than $a'r^n$; but if p is greater than n, then for small values of r at least one of the extreme values of $G_n(x, y)$ is in absolute value smaller than ar^n, however the constant a may be chosen.

It is thus seen that in virtue of the fundamental theorem the function $G_n(x, y)$ may be used as a criterion for determining the existence of a maximum or minimum of the function $f(x, y)$, where $G_n(x, y)$ consists of the terms of the first to the nth order of $f(x, y)$ only when the characteristic exponent p is at most equal to n.

37. If in an example we wished to discuss the function $G_n(x, y)$ in the manner indicated above, we must calculate the coefficients of (*ii*), which, in general, is a somewhat complicated operation. The following method leads, however, indirectly to the same result, viz., that of finding the extreme values of $G_n(x, y)$, and thus offers an easy method for the criteria in question. The method in question is first to make use of the Stolzian theorems of § 29, and then by applying the Scheefferian theorem we may reach the desired conclusions. Accordingly we must determine the upper and lower limits of $G_n(x, y)$ with constant x and $|y| \leqq |x|$ as well

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as the upper and lower limits of this function with constant y and $|x| \leqq |y|$. For brevity put $G = G_n(x, y)$.

The values of y, viz., $y = \Phi_2(x)$ and $y = \Phi_1(x)$, which offer the first-mentioned pair of limits, fall either *within* the interval $-x \cdots +x$ or upon one of the end-values $y = -x$ or $y = +x$. When they fall within the interval, since $G_n(x, y)$ is a continuous function which has a first derivative with respect to y, it is seen that $y = \Phi_1(x)$ and $y = \Phi_2(x)$ are solutions of the equation $\frac{\partial G}{\partial y} = 0$. In the second case, when they fall upon the end-points of the interval, then $y = x$ or $y = -x$ may offer the desired limit or limits.

It is permissible throughout the whole discussion to fix a positive quantity $\alpha < 1$ as the upper limit for $|x|$, where α is taken so small that $y = \Phi_2(x)$ and $y = \Phi_1(x)$ are convergent series in x, which when substituted in the equation $\frac{\partial G}{\partial y} = 0$ identically satisfy it. Furthermore (see § 29), since $\lim_{x=0} \Phi_1(x)$ and $\lim_{x=0} \Phi_2(x) = 0$, it is seen that no constant term can enter these expressions.

The method of determining the different values of y which satisfy the equation $\frac{\partial G}{\partial y} = 0$ is found in §§ 139 et seq. Let these values be

$$P_1(x), \quad P_2(x), \quad P_3(x)_3. \tag{i}$$

38. We may next see which of these functions may be neglected from the investigation. If $P(x)$ denotes any of the functions $P_i(x)\,(i = 1, 2, \cdots)$ and if $P(x)$ has the form

(1) $P(x) = x^\rho \{a + x^\sigma R(x)\}$, where $\rho > 0$ and $\sigma > 0$,

then to any arbitrarily chosen $\epsilon > 0$ there corresponds a quantity $\delta > 0$ such that there are values $|x| < \delta$ for which $|x^\sigma R(x)| < \epsilon$; and for such values of x we have

$$|P(x)| > |x^\rho| \{|a| - \epsilon\}. \tag{2}$$

If ρ lies within the interval $0 < \rho < 1$ and if $|x|$ is further so diminished that $|a| - \epsilon > |x|^{1-\rho}$, then from (2) it is seen that $|P(x)| > |x|$; and consequently $y = P(x)$ would fall without the fixed interval $-x \cdots +x$. We see, therefore, that any series which

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begins with a term $ax^\rho + \cdots$, where $0 < \rho < 1$, may be neglected from the number of functions given in (*i*).

If, next, $\rho = 1$ and $|a| > 1$, we may take ϵ so small in (2) that $|a| - \epsilon > 1$, and consequently $|P(x)| > |x|$, so that such series may also be neglected.

Furthermore, if one of the series (*i*) begins with $+1 \cdot x$ or $-1 \cdot x$, and if the second term has the same sign as the first, then evidently $|P(x)| > |x|$, and such a series may accordingly be neglected from the investigation.

39. The remaining series in (*i*), together with the values which correspond to the end-points, viz., $y = +x$ and $y = -x$, give, when substituted in $G(x, y)$, the following functions:

$$G(x, -x),\ G(x, +x),\ G(x, P_1(x)),\ G(x, P_2(x)),\ \cdots; \qquad (ii)$$

and we have to determine which of these functions presents the upper and the lower limits of the function $G(x, y)$ for the interval in question.

By taking $\alpha (< 1)$ sufficiently small the first term in any of the functions (*ii*) is as a rule sufficient in determining which will give the required upper and lower limits. Of course, if two of the functions (*ii*) have their initial terms the same, it may be necessary to introduce their second and higher terms to determine which furnish the required limits.

Of those functions whose first terms are negative the one with smallest exponent gives the lowest limit; and if two series have the same negative exponent, the one with greater coefficient offers the lower limit. If there is no function in (*ii*) whose first term is negative, then in determining $G(x, \Phi_1(x))$ we note that of those functions whose first terms are positive that one with highest exponent offers the lowest limit; while if two functions have first terms with the same exponent, the one with smaller coefficient offers the lower limit. These observations must be made with both positive and negative values of x, where $|x| < \alpha$. If one of the functions in the series (*ii*) is zero, while the others all begin with a positive term, then $G(x, \Phi_1(x)) = 0$, etc. We proceed in the same way in determining $G(x, \Phi_2(x))$.

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40. To determine $G(\Psi_1(y), y)$ and $G(\Psi_2(y), y)$, taking y constant, we limit x to the interval $-y \cdots +y$. Denote by

$$Q_1(y), \quad Q_2(y), \quad Q_3(y), \quad \cdots,$$

those values of x which expressed in power series in terms of y satisfy the equation $\frac{\partial G}{\partial x} = 0$.

The two limits in question are to be found among the functions

$$G(-y, y), \quad G(y, y), \quad G(Q_1(y), y), \quad G(Q_2(y), y), \quad \cdots,$$

the method of procedure being the same as above.

When each of the four limits $G(\Phi_1(x), x)$, etc. has been determined for values of x within the fixed intervals, the Stolzian theorem is at once applicable. If the expansion, say, of $G(x, \Phi_1(x))$ is $a_k x^k + a_{k+1}x^{k+1} + \cdots$ and if $k \leqq n$, we may at once find a constant c such that $G(x, \Phi_1(x)), \leqq c|x|^n$;

and if the same is true of the three other limits the Scheefferian theorem is at once applicable.

41. Exceptional cases. If the function $G(x, y)$ contains factors, say $x \pm y$, then $G(x, \mp x)$ identically vanishes. More generally the equations $G(x, y) = 0$ and $\frac{\partial G}{\partial y} = 0$ may be satisfied by the same series $y = P(x)$. In this case, considered as an integral function in y and with arbitrary x, the function $G(x, y)$ has a repeated factor, say $Q(x, y)$, which vanishes for $y = P(x)$. Next suppose that $G(x, y)$ is decomposed into its irreducible factors $H_1(x, y)$, $H_2(x, y), \cdots$, and give to x such a value x_1 that each of these functions is also irreducible when considered as a function of y. Furthermore, since by hypothesis $G(x_1, y) = 0$ contains a repeated root $y = P(x_1)$, it is seen that two of the functions $H_1(x_1, y)$, $H_2(x_1, y), \cdots$, say H_1 and H_2, vanish for $y = P(x_1)$. And since by hypothesis these functions are both irreducible with regard to y, they are identical except as to a multiplicative factor which is independent of y. But as $H_1(x, y)$ and $H_2(x, y)$ are identical in y for an indefinitely large number of values such as $x = x_1$, it follows that the coefficients of like powers of y in these two

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functions are identical, so that $G(x, y)$ is divisible at least by the square of an integral function $H(x, y)$.

If at least one of the four functions, say $G(x, \Phi_1(x))$, vanishes for values of x other than $x = 0$ within the fixed intervals, while for all other values this function retains the same sign, and if the other three functions are invariably of this same sign, then $G(0, 0)$ is an *improper* extreme of $G(x, y)$. It follows that as a necessary condition for $G(x, y)$ to have an improper extreme on the position $x = 0$, $y = 0$, $G(x, y)$ must contain as factor the even power of an integral function $H(x, y)$ which not only vanishes for $x = 0$, $y = 0$ but also for values x, y whose absolute values are arbitrarily small. For if, in accordance with the above remarks, $G = H^k \overline{G}$, where $\overline{G}(x, y)$ contains no root $y = P(x)$ which is also contained in $H(x, y)$, and if k is odd, then as y passes through the value $y = P(x)$ the function H^k changes sign and therefore has values with opposite sign.

Example 1. Let $f(x, y) = ay^2 + 2bx^2y + cx^4 + R_5(x, y)$, where $a > 0$ and $R_5(x, y)$ denotes any series beginning with terms of the fifth order in x and y.

Writing $G(x, y) = ay^2 + 2bx^2y + cx^4$, it is seen that for x constant and $|y| \leqq |x|$, $\dfrac{\partial G}{\partial y} = 2(ay + bx^2)$ is zero only for $y = -\dfrac{b}{a}x^2$. We thus have

$$G\left(x, -\frac{b}{a}x^2\right) = \frac{ac - b^2}{a}x^4$$

and

$$G(x, \pm x) = ax^2 \pm 2bx^3 + cx^4.$$

The first expression offers the lower limit, while either $G(x, +x)$ or $G(x, -x)$ offers the upper limit.

We have three cases to consider:

(α) $ac - b^2 < 0$. Then of the two limits one is positive and the other negative. It follows that $G(0, 0)$ is *not* an extreme of $G(x, x)$, and as both limits begin with powers of x not exceeding the fourth, the Scheeffer theorem is applicable, which shows that $f(0, 0)$ is *not* an extreme of $f(x, y)$.

(β) $ac - b^2 > 0$. It follows since $a > 0$ that c must also be positive. The two limits just derived are *both* positive. Continuing we must next determine the other two limits. When y is constant and $|x| \leqq |y|$, we have by solving the equation

$$0 = \frac{\partial G}{\partial x} = 4x(by + cx^2)$$

the two values

$$x = 0 \quad \text{and} \quad x = \pm\sqrt{-\frac{b}{c}y^{\frac{1}{2}}}.$$

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If $b \neq 0$ the latter value may be neglected (§ 38), since the exponent of y lies between 0 and 1. If $b = 0$ this value coincides with the first.

We observe that each of the functions

$$G(0, y) = ay^2 \quad \text{and} \quad G(\pm y, y) = ay^2 + 2by^3 + cy^4$$

is positive. It follows from Stolz's theorem that $G(0, 0)$ is a proper minimum of $G(x, y)$; and since the power of x or y on the right-hand side of any of the four limits is not greater than 4, the Scheeffer theorem shows that $f(0, 0)$ is a proper minimum of $f(x, y)$.

(γ) $ac - b^2 = 0$. From above $G\left(x, -\frac{bx^2}{a}\right) = 0$, while the other three limits are all positive. In this case $G(0, 0)$ is an *improper* minimum and the Scheeffer theorem is *not* applicable, so long as we regard $R_5(x, y)$ as an arbitrary power series with initial term of the fifth or higher dimension. (Stolz, p. 235.)

Example 2. $f(x, y) = y^2 + (ax^2 + 2bxy + cy^2)y + R_4(x, y)$, $(a \neq 0)$.

We have here $\quad G(x, y) = y^2 + (ax^2 + 2bxy + cy^2)y.$

Taking x constant and $|y| \leqq |x|$, we find as a solution of

$$\frac{\partial G}{\partial y} = 0 = 2y + ax^2 + 2bxy + cy^2 + 2y(bx + cy)$$
$$y = -\frac{a}{2}x^2 + \cdots = \phi(x), \text{ say.}$$

Forming the functions

$$G(x, \phi(x)) = -\frac{a^2}{4}x^4 + \cdots \quad \text{and} \quad G(x, \pm x) = x^2 + [2b \pm (a + c)]x^3$$

it is seen that the first furnishes the lower limit, while one of the last functions offers the upper limit. It is evident that with x taken sufficiently small these two limits have contrary signs, so that $G(0, 0)$ is *not* an extreme of $G(x, y)$. Furthermore, since the lower limit begins with a power of x greater than 3, the added theorem of § 31 is *not* applicable.

Proceeding further and taking y constant and $|x| \leqq |y|$, we have as a solution of

$$\frac{\partial G}{\partial x} = 0 = 2y(ax + by) \text{ (since } y \text{ is taken constant)}$$
$$x = -\frac{b}{a}y, \text{ which cannot be considered (§ 38)}$$

unless $|b| < |a|$. Forming the functions

$$G\left(-\frac{b}{a}y, y\right) = y^2 + \cdots; \quad G(\pm y, y) = y^2 + (a \pm 2b + c)y^3,$$

it is seen that both the upper and lower limits are positive. It follows that the added theorem is not applicable. We cannot, therefore, make a negative assertion regarding the extremes of $f(x, y)$. (Stolz.)

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Example 3. $f(x, y) = y^2 + x^2y + x^4 + R_5(x, y)$.

In this example we have $G(x, y) = y^2 + x^2y + x^4$.

With x constant and $|y| \leqq |x|$, we have as the solution of

$$\frac{\partial G}{\partial y} = 0 = 2y + x^2, \quad y = -\frac{x^2}{2}.$$

We thus have the functions

$$G\left(x, -\frac{x^2}{2}\right) = \frac{3}{4}x^4 \quad \text{and} \quad G(x, \pm x) = x^2 + \cdots.$$

With y constant and $|x| \leqq |y|$, we have from

$$\frac{\partial G}{\partial x} = 2xy + 4x^3 = 0, \quad x^2 = -\frac{y}{2}.$$

It follows at once that

$$G\left(i\sqrt{\frac{y}{2}}, y\right) = \frac{3}{4}y^2 \quad \text{and} \quad G(\pm y, y) = y^2 + \cdots.$$

The value $G(0, 0)$ is consequently a proper minimum of $G(x, y)$, and as none of the above series has an initial term with exponent greater than 4, it follows from Scheeffer's theorem that $f(0, 0)$ is a proper minimum of $f(x, y)$. Although there is a proper minimum for $f(x, y) = y^2 + x^2y + x^4$, it may be shown that $G(x, y) = y^2 + x^2y$ has neither a maximum nor a minimum. (Scheeffer, loc. cit., p. 573.)

Example 4. Peano's classic example:

$$f(x, y) = G(x, y) + R_5(x, y),$$

where

$$G = y^2 - (p^2 + q^2)x^2y + p^2q^2x^4.$$

With x constant and $|y| \leqq |x|$, we have

$$\frac{\partial G}{\partial y} = 2y - (p^2 + q^2)x^2 = 0,$$

so that

$$y = \frac{p^2 + q^2}{2}x^2.$$

Forming the functions

$$G\left(x, \frac{p^2 + q^2}{2}x^2\right) = -\frac{x^4}{4}(p^2 - q^2)^2, \quad G(x, \pm x) = x^2 \pm \cdots,$$

it is seen that the upper limit is positive, while the lower limit is negative. It follows that $G(0, 0)$ is *not* an extreme of $G(x, y)$; and as the initial terms on the right have exponents that are not greater than 4, it follows from the Scheeffer theorem that $f(0, 0)$ is *not* an extreme of $f(x, y)$.

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Example 5. $f(x, y) = G(x, y) + R_{13}(x, y)$,

where $G(x, y) = x^2y^4 - 3\,x^4y^3 + (x^6y^2 - 3\,xy^7 + y^8) - 10\,x^{10}y + 5\,x^{12}$.

With x constant and $|y| \leqq |x|$, we have from

$$\frac{\partial G}{\partial y} = 4\,x^2y^3 - 9\,x^4y^2 + (2\,x^6y - 21\,xy^6 + 8\,y^7 - 10\,x^{10}) = 0,$$

as a solution (see § 145),

$$y = 2\,x^2 + \tfrac{5}{7}\,x^4 + \cdots = \phi(x), \text{ say.}$$

Forming the functions

$$G(x, \phi(x)) = -\,4\,x^{10} + \cdots \quad \text{and} \quad G(x, \pm x) = x^6 \pm \cdots,$$

which (see again § 145) offer the upper and lower limits of $G(x, y)$, it follows from Stolz's theorem and the Scheeffer theorem that neither $G(x, y)$ nor $f(x, y)$ has an extreme on the position $x = 0$, $y = 0$. (Scheeffer, loc. cit. p. 575.)

PROBLEMS

1. Show that $f(0, 0)$ is a minimum of

$$f(x, y) = y^4 + x^6 - 108\,x^5y - x^8 + R_9(x, y). \quad \text{(Stolz.)}$$

2. Writing $\quad G(x, y) = y^2 - 2\,x^2y + x^4 + y^4,$

$$f(x, y) = y^2 - 2\,x^2y + x^4 + y^4 - x^6,$$

show that $G(0, 0)$ is a minimum for the first function but that $f(0, 0)$ is *not* a minimum for the second function. Write in the latter expression $y = x^2$. (Scheeffer.)

IV. THE METHOD OF VICTOR VON DANTSCHER

42. Instead of considering the extremes upon the straight lines through the point $P(x_0, y_0)$ we may derive the criteria for maxima and minima in the neighborhood of the points on these lines on both sides of the point (x_0, y_0) in the xy-plane. With Von Dantscher* let the straight lines through (x_0, y_0) be denoted by

$$(1) \qquad x = x_0 + \lambda\rho, \quad y = y_0 + \mu\rho,$$

so that $\quad \dfrac{x - x_0}{\lambda} = \dfrac{y - y_0}{\mu}, \quad \text{or} \quad y - y_0 = \dfrac{\mu}{\lambda}(x - x_0),$

where λ and μ are real variables such that $\lambda^2 + \mu^2 = 1$ and where ρ is a real variable which may have both positive and negative values.

*See *Math. Ann.*, Vol. XLII, p. 89.

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For extremes of $f(x, y)$ at the point P_0 we must have

$$f(x_0+\lambda\rho, y_0+\mu\rho)-f(x_0, y_0) \gtrless 0 \quad \begin{pmatrix} \textit{in case of a maximum} \\ \textit{proper or improper} \end{pmatrix}$$

$$f(x_0+\lambda\rho, y_0+\mu\rho)-f(x_0, y_0) \gtrless 0 \quad \begin{pmatrix} \textit{in case of a minimum} \\ \textit{proper or improper} \end{pmatrix}$$

for all values of ρ of a certain interval

$$-p<\rho<q,$$

while for values $\rho=p$ or $\rho=q$ the above difference not only vanishes but changes sign.

The thesis of Von Dantscher may be stated as follows: "If the lower limit, r say, of ρ in the region $\lambda^2+\mu^2=1$ is different from zero, then $f(x_0, y_0)$ is a maximum or minimum for the surface-neighborhood of the point (x_0, y_0); but if the lower limit of ρ is zero, then on the position x_0, y_0 there is neither a maximum nor a minimum of the function $f(x, y)$."

The decision as to whether a maximum or minimum exists for a given function $f(x, y)$ on a point x_0, y_0 in whose neighborhood $f(x, y)$ can be developed in integral positive powers of $x-x_0=h$, $y-y_0=k$, and on which point the first partial derivatives with respect to x and y both vanish, is consequently reduced to the investigation as to whether the quantity ρ is different from zero or not.

If in the supposed development the nth dimension is the first whose terms do not all vanish, we write

$$(2)\quad f(x_0+h, y_0+k)-f(x_0, y_0) = g(h, k)=(h, k)_n+(h, k)_{n+1}+(h, k)_{n+2}+\cdots, \quad (n \geqq 2)$$

where $(h, k)_n$ denotes the sum of the terms of the nth dimension in h and k, etc.

If we write in this expression

$$(3)\qquad h=\lambda\rho, \quad k=\mu\rho, \quad \lambda^2+\mu^2=1,$$

we have

$$(4)\quad g(h, k)=\rho^n[(\lambda, \mu)_n+\rho(\lambda, \mu)_{n+1}+\cdots]=\rho^n\phi(\rho; \lambda, \mu).$$

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The factor ρ^n may be omitted, since to the value $\rho = 0$ there corresponds the position $h = 0$, $k = 0$ itself. The quantity r is accordingly nothing other than the lower limit of the absolute values of the real roots of the equation

$$(5) \qquad \phi(\rho;\ \lambda,\ \mu) = (\lambda,\ \mu)_n + (\lambda,\ \mu)_{n+1}\rho + \cdots = 0.$$

From this the following is at once evident:

CASE I. If $(h,\ k)_n$ is a *definite form* (§ 13), that is, one which takes the value zero for the one and only pair of values $h = 0$, $k = 0$, which case can only enter when n is even, then $(\lambda,\ \mu)_n$ is different from zero for all values λ, μ which are different from zero, and consequently $|(\lambda,\ \mu)_n|$ has a lower limit l which is different from zero. We may, consequently, for the region $\lambda^2 + \mu^2 = 1$, determine a positive quantity τ such that for $|\rho| < \tau$ we have

$$(\lambda,\ \mu)_n \geqq l > |(\lambda,\ \mu)_{n+1}\rho + (\lambda,\ \mu)_{n+2}\rho^2 + \cdots|.$$

The equation (5) has therefore no root ρ whose absolute value is not greater than τ; the quantity r is therefore different from zero, and *there is consequently a maximum or minimum according as* $(h,\ k)_n$ *is a negative or positive form.*

CASE II. If $(h,\ k)_n$ is an *indefinite form* (§ 13), that is, one which for real pairs of values $(h,\ k)$ takes both positive and negative values, then also $(\lambda,\ \mu)_n$ is such a form. It is then easy to show that in this case the equation

$$0 = (\lambda,\ \mu)_n + (\lambda,\ \mu)_{n+1}\rho + \cdots,$$

in any interval as small as we please $-\epsilon < \rho < \epsilon$, has roots that are different from zero, and consequently $r = 0$ and *also* $f(x_0, y_0)$ *is neither a maximum nor a minimum.*

43. CASE III. We come finally to the *semi-definite case* (§ 13); that is, one where $(h,\ k)_n$ vanishes for pairs of values h, k which are different from zero, but does *not* change sign. It contains necessarily real linear factors, and, in fact, each one to an even power. The number n is consequently even,

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and it follows that $(\lambda, \mu)_n$ is necessarily also a semi-definite form, whose factors are, say,

$$k_1 h - h_1 k,\ k_2 h - h_2 k,\ \cdots,\ k_m h - h_m k, \tag{6}$$

so that $(h, k)_n$ is of the form

$$(h, k)_n = (k_1 h - h_1 k)^{2 l_1} (k_2 h - h_2 k)^{2 l_2} \cdots (k_m h - h_m k)^{2 l_m} (h, k)_{n - 2(l_1 + l_2 + \cdots + l_m)},$$

where $l_1, l_2, \cdots l_m$ are positive integers and $(h, k)_{n-2(l_1+l_2+\cdots+l_m)}$ is a definite form or a constant.

To each such linear factor $k_\sigma h - h_\sigma k (\sigma = 1, 2, \cdots, m)$ of $(h, k)_n$ there corresponds a linear factor $\mu_\sigma \lambda - \lambda_\sigma \mu$ of $(\lambda, \mu)_n$, where

$$\lambda_\sigma = \frac{h_\sigma}{\sqrt{h_\sigma^2 + k_\sigma^2}}, \quad \mu_\sigma = \frac{k_\sigma}{\sqrt{h_\sigma^2 + k_\sigma^2}}, \quad \lambda_\sigma^2 + \mu_\sigma^2 = 1,$$

with arbitrary sign of $\sqrt{h_\sigma^2 + k_\sigma^2}$, since this constant enters only to squared terms in $(h, k)_n$. If λ, μ approach a pair of values λ_σ, μ_σ for which $(\lambda, \mu)_n$ vanishes, then of the roots of the equation

$$\phi(\rho;\ \lambda, \mu) = (\lambda, \mu)_n + (\lambda, \mu)_{n+1} \rho + \cdots = 0,$$

one or several become indefinitely small.

Of course we may exclude the case where all the quantities $(\lambda, \mu)_{n+\nu} (\nu \geqq 1)$ simultaneously vanish; for then $\phi(\rho;\ \lambda, \mu) = 0$ for every arbitrary small value of ρ, and consequently $f(x_0, y_0)$ is neither a maximum nor a minimum.

We have next to see whether among the roots of $\phi(\rho;\ \lambda, \mu) = 0$, which become indefinitely small when $(\lambda, \mu)_n$ becomes indefinitely small, there are *real* roots or not. *If no real roots appear, then* $r > 0$ *and* $f(x_0, y_0)$ *is a maximum if the semi-definite form* $(h, k)_n$, *when it does not vanish, is negative, while it is a minimum if* $(h, k)_n$ *is positive.*

When there appear real roots the investigation may be carried out as follows: In order to consider the function $\phi(\rho;\ \lambda, \mu)$ in the *neighborhood* of the point λ_σ, μ_σ, we write

$$\lambda = \lambda_\sigma + u, \quad \mu = \mu_\sigma + v, \tag{7}$$

where u and v are variable quantities.

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Since $\lambda^2+\mu^2=1$ and $\lambda_\sigma^2+\mu_\sigma^2=1$, we must have

$$u^2+v^2+2\lambda_\sigma u+2\mu_\sigma v=0,$$

where it is certain that one of the quantities λ_σ or μ_σ is different from zero.

If $\mu_\sigma \neq 0$ we have at once from the equation just written

$$v=-\mu_\sigma+\sqrt{\mu_\sigma^2-(2\lambda_\sigma u+u^2)},$$

where the positive sign is taken with the root, since from (i) u and v vanish simultaneously. Further, noting the development

$$\sqrt{\mu_\sigma^2-(2\lambda_\sigma u+u^2)}=\mu_\sigma\sum_{\nu=0}^{\nu=\infty}(-1)^\nu\binom{\frac{1}{2}}{\nu}\frac{(2\lambda_\sigma u+u^2)^\nu}{\mu_\sigma^{2\nu}}$$

it is seen that

$$(8)\qquad v=-\frac{\lambda_\sigma}{\mu_\sigma}u-\frac{1}{2\mu_\sigma^3}u^2-\frac{\lambda_\sigma}{2\mu_\sigma^5}u^3-\cdots;$$

and if $\lambda_\sigma \neq 0$,

$$(9)\qquad u=-\frac{\mu_\sigma}{\lambda_\sigma}v-\frac{1}{2\lambda_\sigma^3}v^2-\cdots.$$

Writing these values in $\phi(\rho;\ \lambda,\ \mu)$, we have

$$\phi_\sigma(u,\rho)=\left(\lambda_\sigma+u,\ \mu_\sigma-\frac{\lambda_\sigma}{\mu_\sigma}u-\cdots\right)_n+\left(\lambda_\sigma+u,\ \mu_\sigma-\frac{\lambda_\sigma}{\mu_\sigma}u-\cdots\right)_{n+1}\rho+\cdots;$$

and

$$\phi_\sigma(v,\rho)=\left(\lambda_\sigma-\frac{\mu_\sigma}{\lambda_\sigma}v-\cdots,\ \mu_\sigma+v\right)_n+\left(\lambda_\sigma-\frac{\mu_\sigma}{\lambda_\sigma}v-\cdots,\ \mu_\sigma+v\right)_{n+1}\rho+\cdots,$$

which for sufficiently small values of $|u|$ and $|\rho|$ or of $|v|$ and $|\rho|$ are certainly convergent and may be arranged in powers of u and ρ or of v and ρ.

Since $(\lambda_\sigma,\ \mu_\sigma)_n=0$, it is seen that $\phi_\sigma(0,\ 0)=0$; the case that $\phi_\sigma(0,\ \rho)$ vanishes identically may be excluded, as has already been remarked.

If ρ^p is the lowest power of ρ in $\phi_\sigma(0,\rho)$, the equation $\phi_\sigma(0,\rho)=0$ has exactly p roots ρ, which become indefinitely small with u or v. We must next see whether there are real roots among these p roots.

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If the equation $\phi_\sigma(u, \rho) = 0$ has no real root ρ which becomes indefinitely small with u or v, then for any arbitrarily small positive quantity ϵ a positive quantity δ *cannot* be found so small that in the interval $-\epsilon < \rho < \epsilon$ there is situated a root ρ of $\phi_\sigma(u, \rho) = 0$ or of $\phi_\sigma(v, \rho)$ which is different from zero and which belongs to a value u or v in the interval $-\delta < u < \delta$ or $-\delta < v < \delta$. Hence there exist positive quantities δ and ϵ so small that the function ϕ_σ which vanishes simultaneously with u and ρ or with v and ρ in the region

$$-\delta < u < \delta \quad \text{or} \quad -\delta < v < \delta, \quad -\epsilon < \rho < \epsilon$$

takes values that are different from zero on every position u, ρ or v, ρ which is different from 0, 0, and these values have necessarily the same sign. For if $\phi_\sigma(u', \rho') > 0$ and $\phi_\sigma(u'', \rho'') < 0$, then with a continuous passage from the position u', ρ' to the position u'', ρ'', which both lie within the interior of the realm in question and which passage does not pass through the position 0, 0, there must be a position u_0, ρ_0 at which $\phi_\sigma(u, \rho)$ vanishes; but there are no such positions. It follows that $\phi_\sigma(0, 0)$ is itself a maximum or minimum provided the equation $\phi_\sigma(u, \rho) = 0$ has no real root which becomes simultaneously indefinitely small with u or v. Inversely, it is also true that if $\phi_\sigma(0, 0)$ is a maximum or minimum of $\phi_\sigma(u, \rho)$, the equation $\phi_\sigma(u, \rho) = 0$ has *no* real root which becomes indefinitely small with u or v.

If, on the other hand, the equation $\phi_\sigma(u, \rho) = 0$ has real roots which become indefinitely small with u or v, then $\phi_\sigma(0, 0)$ is neither a maximum nor a minimum; and vice versa, if $\phi_\sigma(0, 0)$ is *not* a maximum or minimum, then in every region as small as we wish $-\delta < u < \delta$ or $-\delta < v < \delta$, $-\epsilon < \rho < \epsilon$ there are positions u, ρ or v, ρ which are different from zero and for which $\phi_\sigma(u, \rho)$ or $\phi_\sigma(v, \rho)$ are zero.

Through the above consideration the criterion whether the equation $\phi_\sigma = 0$ has or has not real roots which become indefinitely small with u or v is reduced to the investigation whether $\phi_\sigma(0, 0)$ is a maximum or minimum of $\phi_\sigma(u, \rho)$ or $\phi_\sigma(v, \rho)$ or not.

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We have, therefore, to apply the criteria of Cases I and II of § 42; that is, to arrange ϕ_σ in dimensions of u and ρ or of v and ρ and to see whether the terms of lowest dimension form a definite or indefinite form.

This same process must be applied to each of the m real linear factors $\mu_\sigma\lambda - \lambda_\sigma\mu (\sigma = 1, 2, \cdots, m)$ that are different from one another (p. 65), it being evidently sufficient, since u and v become simultaneously indefinitely small, for those linear factors in which λ_σ and μ_σ are both different from zero to consider only one of the functions $\phi_\sigma(u, \rho)$ or $\phi_\sigma(v, \rho)$.

44. We have, then, the following rule for Case III:

If the developments of the functions $\phi_1(u, \rho), \phi_2(u, \rho), \cdots, \phi_m(u, \rho)$ *all begin with definite forms, then* $f(x_0, y_0)$ *is a maximum when the semi-definite form* $(h, k)_n \leqq 0$, *while it is a minimum if* $(h, k)_n \geqq 0$.

If only one of the functions $\phi_\sigma(u, \rho)$ *begins with an indefinite form, then* $f(x_0, y_0)$ *is neither a maximum nor a minimum.*

The case remains undetermined if among all the functions $\phi_\sigma(u, \rho)$ *none of them begins with an indefinite form, while one or several of them begin with a semi-definite form.*

In this case, for every such function the above process must be again applied. We do *not* affirm that by using this method a determination may among all conditions be made; but Von Dantscher says "if the method, which has been developed to see whether a series $g(h, k)$ which begins with a semi-definite form has or has not on the position $h = 0$, $k = 0$ a maximum or minimum, fails, the function $g(h, k)$ contains an even power of a series $P(h, k)$ which vanishes for real pairs of values h, k in every region arbitrarily small $0 < |h| < \delta$, $0 < |k| < \delta$" (see § 41).

Example 1. Peano's classic example:

$$g(h, k) = k^2 - (p^2 + q^2)h^2k + p^2q^2h^4.$$

We have here

$$\phi(\rho; \lambda, \mu) = \mu^2 - (p^2 + q^2)\lambda^2\mu\rho + p^2q^2\lambda^4\rho^2.$$

The semi-definite form μ^2 has the linear factor μ so that either

$$\lambda_1 = 1, \quad \mu_1 = 0, \quad \text{or} \quad \lambda_1 = -1, \quad \mu_1 = 0.$$

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The corresponding values of λ and μ are (see [7] and [9])

$$\lambda = 1 - \tfrac{1}{2}v^2 - \cdots, \quad \mu = v,$$

so that $$\phi_1(v, \rho) = v^2 - (p^2 + q^2)v\rho + p^2q^2\rho^2 + ((v, \rho))_3 + \cdots.$$

The terms of the second dimension in v and ρ form an indefinite quadratic form, so that $g(0, 0)$ is neither a maximum nor a minimum.

Example 2. Let $$g(h, k) = -h^2k^2(h - k)^2 + 2hk^6 - 5h^2k^5 + 3h^3k^4 + h^4k^3 - 7h^5k^2 + 6h^6k - 10h^8 + h^2k^6 + 3h^4k^4 - 4k^8 + \cdots.$$

We then have

$$\phi(\rho; \lambda, \mu) = -\lambda^2\mu^2(\lambda - \mu)^2 + (2\lambda\mu^6 - 5\lambda^2\mu^5 + 3\lambda^3\mu^4 + \lambda^4\mu^3 - 7\lambda^5\mu^2 + 6\lambda^6\mu)\rho + (-10\lambda^8 + \lambda^2\mu^6 + 3\lambda^4\mu^4 - 4\mu^8)\rho^2 + \cdots.$$

We have here to consider the three linear factors

$$\mu_1\lambda - \lambda_1\mu = \lambda, \quad \mu_2\lambda - \lambda_2\mu = \mu, \quad \mu_3\lambda - \lambda_3\mu = \lambda - \mu.$$

It follows that

$$\lambda_1 = 0, \quad \mu_1 = 1; \quad \lambda_2 = -1, \quad \mu_2 = 0; \quad \lambda_3 = \frac{1}{\sqrt{2}}, \quad \mu_3 = \frac{1}{\sqrt{2}}.$$

To these values correspond the expressions:

$$\lambda = u, \quad \mu = 1 - \tfrac{1}{2}u^2 - \cdots,$$
$$\mu = v, \quad \lambda = -1 + \tfrac{1}{2}v^2 + \cdots,$$
$$\lambda = \frac{1}{\sqrt{2}} + u, \quad \mu = \frac{1}{\sqrt{2}} - u - \cdots.$$

We thus have

$$\phi_1(u, \rho) = -u^2 + 2u\rho - 4\rho^2 + \cdots,$$
$$\phi_2(v, \rho) = -v^2 + 6v\rho - 10\rho^2 + \cdots,$$
$$\phi_3(u, \rho) = -u^2 + \tfrac{3}{2}u\rho - \tfrac{5}{8}\rho^2 + \cdots.$$

It is seen that all three of the functions ϕ begin with definite quadratic forms. The semi-definite initial form is negative when it is not zero; and accordingly $g(0, 0)$ is a maximum. (Von Dantscher, *Math. Ann.*, Vol. XLII, p. 100.)

PROBLEMS

1. Show that $g(0, 0)$ is neither a maximum nor a minimum of the function

$$g(h, k) = h^2k^4 - 3h^4k^3 + h^6k^2 - 3hk^7 + k^8 - 10h^{10}k + 5h^{12}.$$

(See Ex. 5, p. 62.)

2. Apply this method to Ex. 3, p. 61.

3. If $z^2 = a^2 - x^2 - y^2 + (x\cos\alpha + y\sin\alpha)^2$, find maximum and minimum values of z and give the geometric interpretation.

4. If $z^2 = 2a\sqrt{x^2 + y^2} - x^2 + y^2$, find maximum and minimum values of z; show that there are improper extremes and give geometric signification.

5. Find minimum value of u, where $u = (x^2 + y^2)^{\frac{1}{3}}$.

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V. FUNCTIONS OF THREE VARIABLES

Treatment in Particular of the Semi-definite Case

45. The theorems and proofs given by Stolz and Scheeffer for functions of two variables may be extended at once to functions of three or more variables. For example, $f(x_0, y_0, z_0)$ is a proper maximum of $f(x, y, z)$ if a positive quantity δ can be so determined that for all systems of values ξ, η, ζ, whose absolute values are smaller than δ (excepting $\xi = 0 = \eta = \zeta$), we have

$$f(x_0 + \xi, y_0 + \eta, z_0 + \zeta) - f(x_0, y_0, z_0) < 0.$$

If the partial derivatives of $f(x, y, z)$ have definite values at every position of a fixed realm R, the coördinates x_0, y_0, z_0 of those positions (if any) in R which offer extremes of the function $f(x, y, z)$ must satisfy the equations

$$\left[\frac{\partial f}{\partial x}\right]_0 = 0, \quad \left[\frac{\partial f}{\partial y}\right]_0 = 0, \quad \left[\frac{\partial f}{\partial z}\right]_0 = 0.$$

To apply the Stolzian theorem we observe, if we limit ourselves to a position $x_0 = 0 = y_0 = z_0$, that the collectivity of positions x, y, z for which $|x|, |y|, |z|$ are less than δ are distributed into three kinds of realms:*

(1) always with $|x| < \delta$, x constant, and

$$|y| \leqq |x|, \quad |z| \leqq |x|;$$

(2) with y constant and $|y| < \delta$, where also

$$|x| \leqq |y|, \quad |z| \leqq |y|;$$

(3) with z constant and $|z| < \delta$ and

$$|x| \leqq |z|, \quad |y| \leqq |z|.$$

To apply the Scheeffer theorem we must consider the difference

$$f(x, y, z) - f(0, 0, 0) = G_n(x, y, z) + R_{n+1}(x, y, z),$$

as in § 30.

The case where $G_n(x, y, z)$ is a definite or indefinite form is treated fully in Chapter V.

* Stolz, loc. cit., p. 237.

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46. The case where $G_n(x, y, z)$ is a nonhomogeneous function in which the terms of the lowest dimension constitute a semi-definite form may be treated in a manner analogous to that given in §§ 37–41, as follows:

We first determine the upper and lower limits of $G(x, y, z)$ with constant x and $|y| \leqq |x|$, $|z| \leqq |x|$. Geometrically interpreted, this realm constitutes a square whose center is the origin and whose sides are parallel to the y-axis and the z-axis, the length being $2|x|$.

The positions at which $G(x, y, z)$ reaches one of its limits may lie (1) on the vertices, or (2) on the sides, or (3) within the interior of the square.

We have, consequently, to form the expressions corresponding to the four vertices

$$G(x, x, x),\ G(x, x, -x),\ G(x, -x, x),\ G(x, -x, -x). \qquad (i)$$

For points on the sides we have to solve for y the equations

$$(\alpha) \qquad \frac{\partial G(x, y, x)}{\partial y} = 0 \quad \text{and} \quad \frac{\partial G(x, y, -x)}{\partial y} = 0,$$

and for z the equations

$$(\beta) \qquad \frac{\partial G(x, x, z)}{\partial z} = 0 \quad \text{and} \quad \frac{\partial G(x, -x, z)}{\partial z} = 0.$$

Let the solutions of the equations (α) be

$$y = P_1(x), \quad y = P_2(x), \quad \cdots$$

and let the solutions of (β) be

$$z = Q_1(x), \quad z = Q_2(x), \quad \cdots.$$

Those functions $P(x)$ and $Q(x)$ which cause y and z to fall without the given square are to be neglected (cf. § 38).

With the remaining functions we form the expressions

$$G(x, P_1(x), x), \cdots; \quad G(x, x, Q_1(x)). \qquad (ii)$$

For the points *within* the square we have to determine y and z in terms of x from the equations

$$(\gamma) \qquad \frac{\partial G(x, y, z)}{\partial y} = 0 \quad \text{and} \quad \frac{\partial G(x, y, z)}{\partial z} = 0.$$

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If we eliminate z from these two equations, we may express y as power series in x without constant term, say $y = \phi_1(x)$, $y = \phi_2(x)$, $\cdots$ (§ 29). To each such power series for y, say $y = \phi(x)$, there corresponds one for z in terms of x, say $z = \lambda(x)$, which two series written in the two equations (γ) cause them to vanish identically. With these values of y and z we form the expressions

$$G(x, \phi_1(x), \lambda_1(x)), \quad G(x, \phi_2(x), \lambda_2(x)). \qquad (iii)$$

Among all the functions that are found in (*i*), (*ii*), and (*iii*) we are now able to determine those two which offer the upper and lower limits of the function $G(x, y, z)$ within the interval in question. These limits may be denoted by $G(x, \Phi_2(x), \Lambda_2(x))$ and $G(x, \Phi_1(x), \Lambda_1(x))$.

If, next, we take y constant and $|x| \leqq |y|$, $|z| \leqq |y|$, we may derive in a similar manner the upper and lower limits $G(\Psi_2(y), y, \mathrm{M}_2(y))$ and $G(\Psi_1(y), y, \mathrm{M}_1(y))$. Finally, with z constant and $|x| \leqq |z|$, $|y| \leqq |z|$, we derive the upper and lower limits

$$G(\mathrm{N}_2(z), \Omega_2(z), z) \quad \text{and} \quad G(\mathrm{N}_1(z), \Omega_1(z), z).$$

The Stolzian and Scheefferian theorems are at once applicable to these six functions in three variables, the method of procedure being an evident generalization of these theorems for the functions in two variables.

PROBLEMS

1. Make the extension and generalization of Von Dantscher's method to the treatment of functions in three variables.

2. In the line of intersection of two given planes find the nearest point to the origin of coördinates.

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CHAPTER V

MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES THAT ARE SUBJECTED TO NO SUBSIDIARY CONDITIONS

I. ORDINARY EXTREMES

47. It will be presupposed in the following discussion, unless it is expressly stated to the contrary, that not only the quantities that appear as arguments of the functions but also the functions themselves are real, and that the functions, as soon as the variables are limited to a definite continuous region, have within this region everywhere the character of one-valued regular functions. *Regular functions* are defined in the following manner: *A function $f(x)$ is* ***regular*** *within certain fixed limits of x if the function is defined for all values of x within these limits and if for every value a of x within these limits the development*

$$f(a+h)=f(a)+\frac{h}{1!}f'(a)+\frac{h^2}{2!}f''(a)+\cdots$$

is possible; the series must be convergent and must in reality (see § 136), *represent the values of the function within this neighborhood.*

In other words: *A function $f(x)$ is* ***regular*** *in the neighborhood of the position $x=a$ if the function in this neighborhood has everywhere a definite value which changes in a continuous manner with x.* (Cf. Weierstrass, *Werke*, Vol. II, p. 77.)

A one-valued analytic function $f(x_1, x_2, \cdots, x_n)$ of several variables behaves ***regularly*** *on a definite position $(x_1=a_1, x_2=a_2, \cdots, x_n=a_n)$ if in the neighborhood of this position we may express the function through a series of the form*

$$\sum A_{\nu_1, \nu_2, \cdots, \nu_n}(x_1-a_1)^{\nu_1}(x_2-a_2)^{\nu_2}\cdots(x_n-a_n)^{\nu_n},$$

where $\nu_1, \nu_2, \cdots, \nu_n$ are positive integers or zero, and where the coefficients $A_{\nu_1, \nu_2, \cdots, \nu_n}$ are quantities that are independent of the variables. (Cf. Weierstrass, *Werke*, Vol. II, p. 154.)

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The discussion is thus limited to such functions as are analytic structures of the nature described more in detail in §§ 130, 131. Only for such functions can we derive general theorems, since for other functions even the rules of the differential calculus are not applicable; in other words we shall consider only the *ordinary* extremes.

The problem of finding those values of the argument of a function $f(x)$ for which the function has a maximum or minimum value is not susceptible of a *general* solution, for, besides the cases of the extraordinary extremes of §§ 5–7, there are functions which, in spite of the fact that they may be defined through a simple series or through other algebraic expressions and which vary in a continuous manner, have an infinite number of maxima and minima within an interval which may be taken as small as we wish.* Such functions do not come under the present investigation.

48. We say (see § 1) that *a function $f(x)$ of one variable has a proper maximum or a proper minimum at a definite position $x = a$ if the value of the function for $x = a$ is respectively greater or less than it is for all other values of x which are situated in a neighborhood $|x - a| < \delta$ as near as we wish to a.*

The analytical condition that $f(x)$ shall have for the position $x = a$

$$\left.\begin{array}{l} a \text{ proper maximum, is expressed by } f(x)-f(a)<0 \\ a \text{ proper minimum, is expressed by } f(x)-f(a)>0 \end{array}\right\} \text{ for } |x-a|<\delta.$$

In the same way we say *a function $f(x_1, x_2, \cdots, x_n)$* of *n variables has at a definite position $x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n$ a proper maximum or a proper minimum if the value of the function for $x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n$ is respectively greater*

* A function may have in an interval as small as we wish

(1) an infinite number of discontinuities,
(2) an infinite number of maxima and minima,

and still be expressed through a Fourier series.

See, for example, H. Hankel, *Ueber die unendlich oft oscillirenden und unstetigen Functionen* (Tübingen, 1870); Lipschitz, *Crelle*, Vol. LXIII, p. 296; P. du Bois-Reymond, *Abh. der Bayer. Akad.*, Vol. XII, p. 8, and also same volume, Part II, *Math.-Phys. Classe* (1876).

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or less than it is for all other systems of values situated in a neighborhood

$$|x_\lambda - a_\lambda| < \delta_\lambda \quad (\lambda = 1, 2, \cdots, n)$$

as near as we wish to the first position; and the analytical condition that the function $f(x_1, x_2, \cdots, x_n)$ shall have at the position $x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n$

a proper maximum, is $f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n) < 0$,

a proper minimum, is $f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n) > 0$,

for $|x_\lambda - a_\lambda| < \delta_\lambda$ $(\lambda = 1, 2, \cdots, n)$, where the quantities δ_λ are arbitrarily small. Improper extremes take the place of the proper extremes above when we allow the equality sign to appear with the inequality sign, as in § 1.

49. The problem which we have to consider in the theory of maxima and minima is, then, to find those positions at which a maximum or minimum really enters.

We shall give a brief résumé of this problem for functions of one variable and then make its generalization for functions of several variables.

If x_1, x_2 are two values of x situated sufficiently near each other within a given region, then the difference of the corresponding values of the function is expressible in the form:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1 + \theta(x_2 - x_1)),$$

where θ denotes a quantity situated between 0 and 1; or, if x_1 is written equal to x and $x_2 = x + h$,

$$[1] \qquad f(x+h) - f(x) = hf'(x + \theta h).$$

From this theorem may be derived Taylor's theorem in the form,*

$$[2] \qquad f(x+h) - f(x) = hf'(x) + \frac{1}{2!}h^2 f''(x) + \cdots + \frac{1}{(n-1)!}h^{n-1}f^{(n-1)}(x) + \frac{1}{n!}h^n f^{(n)}(x + \theta h).$$

* See Jordan, *Cours D'Analyse*, Vol. I, §§ 249–250.

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In the two formulæ last written, instead of $x+h$ write x and write a in the place of x; they then become

$$[1^a] \qquad f(x)-f(a)=(x-a)f'(a+\theta(x-a))$$

and

$$[2^a] \qquad f(x)-f(a)=\frac{x-a}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$$
$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(x+\theta(x-a)).$$

Since $f(x)$ is a regular, and consequently continuous, function, the same is true of all its derivatives. If $f'(a)$ is different from zero, then with small values of $h=x-a$ the value of $f'(a+\theta h)$ is different from zero and has the same sign as $f'(a)$.

According to the choice of h, which is arbitrary, the difference $f(x)-f(a)$ can be made to have one sign or the opposite sign, if $f'(a)$ is either a positive quantity or a negative quantity. Hence the function $f(x)$ can have neither a maximum nor a minimum value at the position $x=a$ if $f'(a)\leqq 0$.

We therefore have the theorem: *Extremes of the function $f(x)$ can only enter for those values of x for which $f'(x)$ vanishes* (see § 2).

It may happen that for the roots of the equation $f'(x)=0$ some of the following derivatives also vanish. If the nth derivative is the first one that does not vanish for the root $x=a$, then from equation $[2^a]$ we have the formula

$$f(x)-f(a)=\frac{(x-a)^n}{n!}f^{(n)}(x+\theta(x-a)),$$

and with small values of $h=x-a$, owing to the continuity of $f^{(n)}(x)$, the quantity $f^{(n)}(a+\theta h)$ will likewise be different from zero and will have the same sign as $f^{(n)}(a)$. If, therefore, n is an odd integer, we may always bring it about, according as h is taken positive or negative, that the difference $f(x)-f(a)$ with every value of $f^{(n)}(a)$ has either one sign or the opposite sign; consequently the function $f(x)$ will have at the position $x=a$ neither a maximum nor a minimum value.

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If, however, n is an even integer, then h^n is always positive, whatever the choice of h may have been; consequently the difference $f(x)-f(a)$ is positive or negative according as $f^{(n)}(a)$ is positive or negative.

In the first case the function $f(x)$ has a minimum value at the position $x=a$; in the latter case, a maximum.

Taking this into consideration we have the following theorem for functions of one variable (§ 3):

Extremes of the function $f(x)$ can only enter for the roots of the equation $f'(x)=0$. If a is a root of this equation, then at the position $x=a$ there is neither a maximum nor a minimum if the first of the derivatives that does not vanish for this value is of an odd degree; if, however, the degree is even, then the function has a maximum value for the position $x=a$ if the derivative for $x=a$ is negative, a minimum if it is positive.

50. To derive the analog for functions of several variables, we start again with the Taylor-Lagrange theorem * for such functions. This theorem may be derived by first writing in $f(x_1, x_2, \cdots, x_n)$

$$x_\lambda = a_\lambda + u(x_\lambda - a_\lambda), \quad (\lambda = 1, 2, \cdots, n),$$

where u is a quantity that varies between 0 and 1; we then apply to the function

$$\phi(u) = f\big(a_1 + u(x_1 - a_1),\ a_2 + u(x_2 - a_2),\ \cdots,\ a_n + u(x_n - a_n)\big)$$

Maclaurin's theorem for functions of one variable, viz.:

$$[3] \qquad \phi(u) = \phi(0) + \frac{u}{1!}\phi'(0) + \frac{u^2}{2!}\phi''(0) + \cdots + \frac{u^{m-1}}{(m-1)!}\phi^{(m-1)}(0) + \frac{u^m}{m!}\phi^{(m)}(\theta u),$$

and, finally, in this expression write $u=1$, as follows:

For brevity denote by $f_k(x_1, x_2, \cdots, x_n)$ the first derivative of $f(x_1, x_2, \cdots, x_n)$ with respect to x_k and by $f_{k_1, k_2}(x_1, x_2, \cdots, x_n)$ the derivative of $f(x_1, x_2, \cdots, x_n)$ with respect to x_{k_1} and x_{k_2}, that is,

$$f_{k_1, k_2}(x_1, x_2, \cdots, x_n) = \frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_{k_1} \partial x_{k_2}}, \text{ etc.}$$

* See Lagrange, *Théorie des Fonctions*, p. 152.

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It follows, then, that

$$\phi'(u)=\sum_{k=1}^{k=n}\{f_k\big(a_1+u(x_1-a_1),\cdots,a_n+u(x_n-a_n)\big)(x_k-a_k)\},$$

$$\phi''(u)=\sum_{k_1,k_2}\{f_{k_1,k_2}\big(a_1+u(x_1-a_1),\cdots, a_n+u(x_n-a_n)\big)(x_{k_1}-a_{k_1})(x_{k_2}-a_{k_2})\},$$

. . . . . .

$$\phi^{(m-1)}(u)=\sum_{k_1,k_2,\cdots,k_{m-1}}\{f_{k_1,k_2,\cdots,k_{m-1}}\big(a_1+u(x_1-a_1),\cdots, a_n+u(x_n-a_n)\big)(x_{k_1}-a_{k_1})\cdots(x_{k_{m-1}}-a_{k_{m-1}})\},$$

$$\phi^{(m)}(\theta u)=\sum_{k_1,k_2,\cdots,k_m}\{f_{k_1,k_2,\cdots,k_m}\big(a_1+\theta u(x_1-a_1),\cdots, a_n+\theta u(x_n-a_n)\big)(x_{k_1}-a_{k_1})\cdots(x_{k_m}-a_{k_m})\}.$$

Hence, from [3] we have

$$\phi(u)-f(a_1,a_2,\cdots,a_n)=\frac{u}{1}\sum_k\{f_k(a_1,a_2,\cdots,a_n)(x_k-a_k)\}$$
$$+\frac{u^2}{2!}\sum_{k_1,k_2}\{f_{k_1,k_2}(a_1,a_2,\cdots,a_n)(x_{k_1}-a_{k_1})(x_{k_2}-a_{k_2})\}$$
$$+\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$
$$+\frac{u^{m-1}}{(m-1)!}\sum_{k_1,k_2,\cdots,k_{m-1}}\{f_{k_1,k_2,\cdots,k_{m-1}}(a_1,a_2,\cdots,a_n)(x_{k_1}-a_{k_1})\cdots(x_{k_{m-1}}-a_{k_{m-1}})\}$$
$$+\frac{u^m}{m!}\sum_{k_1,k_2,\cdots,k_m}\{f_{k_1,k_2,\cdots,k_m}\big(a_1+\theta u(x_1-a_1),\cdots, a_n+\theta u(x_n-a_n)\big)(x_{k_1}-a_{k_1})\cdots(x_{k_m}-a_{k_m})\}$$

From this it follows, if we write $u=1$, that

$$f(x_1,x_2,\cdots,x_n)-f(a_1,a_2,\cdots,a_n)=\sum_k\{f_k(a_1,a_2,\cdots,a_n)(x_k-a_k)\}$$
$$+\frac{1}{2!}\sum_{k_1,k_2}\{f_{k_1,k_2}(a_1,a_2,\cdots,a_n)(x_{k_1}-a_{k_1})(x_{k_2}-a_{k_2})\}$$
$$+\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$
$$+\frac{1}{(m-1)!}\sum_{k_1,k_2,\cdots,k_{m-1}}\{f_{k_1,k_2,\cdots,k_{m-1}}(a_1,a_2,\cdots,a_n)(x_{k_1}-a_{k_1})\cdots(x_{k_{m-1}}-a_{k_{m-1}})\}$$
$$+\frac{1}{m!}\sum_{k_1,k_2,\cdots,k_m}\{f_{k_1,k_2,\cdots,k_m}\big(a_1+\theta(x_1-a_1),a_2+\theta(x_2-a_2),\cdots, a_n+\theta(x_n-a_n)\big)(x_{k_1}-a_{k_1})\cdots(x_{k_m}-a_{k_m})\}.$$

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51. We are not accustomed to Taylor's theorem* in the form just given; to derive this theorem as it is usually given, observe that upon performing the indicated summations each of the indices $k_1, k_2, \cdots$, independently the one from the other, takes all values from 1 to n, so that the λth term in the development is a homogeneous function of the λth degree in $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$. The general term of this homogeneous function may be written in the form

$$\frac{1}{\lambda!} D \cdot N \cdot (x_1 - a_1)^{\lambda_1} (x_2 - a_2)^{\lambda_2} \cdots (x_n - a_n)^{\lambda_n},$$

where $$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \lambda,$$

D is the definite differential quotient

$$\left(\frac{\partial^\lambda f(x_1, x_2, \cdots, x_n)}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \cdots \partial x_n^{\lambda_n}} \right)_{a_1, a_2, \cdots, a_n} = f^{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}(a_1, a_2, \cdots, a_n),$$

and N is the number of permutations of λ elements of which λ_1, $\lambda_2, \cdots, \lambda_n$ respectively are alike; that is,

$$N = \frac{\lambda!}{\lambda_1!\ \lambda_2!\ \cdots \lambda_n!}.$$

Furthermore, writing $x_k - a_k = h_k$, we have, finally,

$$[4] \quad f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n)$$

$$= \sum_{k=1}^{k=n} \left\{ \left(\frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial x_k} \right)_{a_1, a_2, \cdots, a_n} h_k \right\}$$

$$+ \frac{1}{2} \sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, a_2, \cdots, a_n} h_\lambda h_\mu \right\}$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+ \sum_{\lambda_1, \lambda_2, \cdots, \lambda_n}^{\lambda_1 + \lambda_2 + \cdots + \lambda_n = m-1} \left\{ f^{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}(a_1, a_2, \cdots, a_n) \frac{h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n}}{\lambda_1!\ \lambda_2!\ \cdots \lambda_n!} \right\}$$

$$+ \sum_{\lambda_1, \lambda_2, \cdots, \lambda_n}^{\lambda_1 + \lambda_2 + \cdots + \lambda_n = m} \left\{ f^{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}(a_1 + \theta h_1, \cdots, a_n + \theta h_n) \frac{h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n}}{\lambda_1!\ \lambda_2! \cdots \lambda_n!} \right\}.$$

* Stolz (*Grundzüge der Differential und Integralrechnung*, p. 247) ascribes this mode of expression to A. Mayer (see paper by him in the *Leipz. Ber.* (1889), p. 128). The form as presented here is found in Weierstrass's lectures delivered at least ten years before the Mayer paper.

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This is the usual form of Taylor's theorem for functions of several variables. In particular, when $m=1$ the above development is

$$[5]\quad f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n) = \sum_{k=1}^{k=n} \{f_k(a_1+\theta h_1, \cdots, a_n+\theta h_n) h_k\}.$$

The function $f(x_1, \cdots, x_n)$ is regular and continuous, as are consequently all its derivatives. If, therefore, the first derivatives of $f(x_1, x_2, \cdots, x_n)$ are all, or in part, $\gtrless 0$ for $x_1=a_1, \cdots, x_n=a_n$, then they will also be different from zero for $x_1=a_1+\theta h_1, \cdots, x_n=a_n+\theta h_n$, where the absolute values of $h_1, h_2, \cdots, h_n$ have been taken sufficiently small; these derivatives will also be of the same sign as they were for $x_1=a_1, x_2=a_2, \cdots, x_n=a_n$. If, now, we choose all the h's zero with the exception of one, which may be taken either positive or negative, it is seen that when the corresponding derivative has either sign, we may always bring it about at pleasure that the difference

$$f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n)$$

is either a positive or a negative quantity, and consequently at the position $a_1, a_2, \cdots, a_n$ no extreme value of the function is permissible.

We therefore have the following theorem:

Extremes of the function $f(x_1, x_2, \cdots, x_n)$ can only enter for those systems of values of $(x_1, x_2, \cdots, x_n)$ which at the same time satisfy the n equations (p. 17)

$$[6]\qquad \frac{\partial f}{\partial x_1}=0, \quad \frac{\partial f}{\partial x_2}=0, \quad \cdots, \quad \frac{\partial f}{\partial x_n}=0.$$

It may happen that for the common roots of the system of equations [6] still higher derivatives also vanish. In this case we can in general only say that if for a system of roots of the equations [6] all the derivatives of several of the next higher orders vanish, and if the first derivative which does not vanish for these values is of an odd order, the function, as may be shown by a method of reasoning similar to that above, has certainly no maximum or minimum value.

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52. If, however, this derivative is of an even order, then in the present state of the theory of forms of the nth order in several variables *there is no general criterion* regarding the behavior of the function at the position in question. *We therefore limit ourselves to the case where the derivatives of the second order of the function* $f(x_1, x_2, \cdots, x_n)$ *do not all vanish for the system of real roots* $a_1, a_2, \cdots, a_n$ *of the equations* [6].

In this case we have a criterion in the formula

$$[7] \quad f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n) = \frac{1}{2} \sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \theta h_1, \cdots, a_n + \theta h_n} h_\lambda h_\mu \right\}$$

by which we may determine whether $f(x_1, x_2, \cdots, x_n)$ has an extreme value on the position $a_1, a_2, \cdots, a_n$, since we may determine whether the integral homogeneous function of the second degree,

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \theta h_1, \cdots, a_n + \theta h_n} h_\lambda h_\mu \right\},$$

in the n variables $h_1, h_2, \cdots, h_n$ is for arbitrary values of those variables invariably positive or invariably negative.

Denote this function by $f_{\lambda\mu}(a_1 + \theta h_1, \cdots, a_n + \theta h_n)$.

On account of their presupposed continuity the quantities

$$\left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \theta h_1, \cdots, a_n + \theta h_n} \text{ and } \left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, \cdots, a_n}$$

with values of $h_1, h_2, \cdots, h_n$ taken sufficiently small differ from each other as little as we wish and are of the same sign;* hence with small values of the h's the functions

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \theta h_1, \cdots, a_n + \theta h_n} h_\lambda h_\mu \right\}$$

and

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, \cdots, a_n} h_\lambda h_\mu \right\}$$

have always the same sign, and we may therefore confine ourselves to the investigation of the latter function.

* If any of the quantities $\left(\frac{\partial^2 f(x_1, \cdots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, \cdots, a_n}$ becomes zero, we may replace it by $\epsilon_{\lambda\mu}(a_1, \cdots, a_n)$, which must of course be given the same sign as $f_{\lambda\mu}(a_1 + \theta h_1, \cdots, a_n + \theta h_n)$, $\epsilon_{\lambda\mu}$ denoting an infinitesimally small quantity.

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If it is found that through a suitable choice of $h_1, h_2, \cdots, h_n$ the expression

$$\sum_{\lambda,\mu}\left\{\left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu}\right)_{a_1, a_2, \cdots, a_n} h_\lambda h_\mu\right\}$$

can be made at pleasure either positive or negative, the same will be the case with the difference $f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n)$, and consequently $f(x_1, x_2, \cdots, x_n)$ has on the position $(a_1, a_2, \cdots, a_n)$ no extreme value.

We therefore have as a second condition for the existence of a maximum or a minimum of the function $f(x_1, x_2, \cdots, x_n)$ on the position $(a_1, a_2, \cdots, a_n)$ that in case the second derivatives of the function $f(x_1, x_2, \cdots, x_n)$ do not all vanish at this position, the homogeneous quadratic form

$$\sum_{\lambda,\mu}\left\{\left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu}\right)_{a_1, a_2, \cdots, a_n} h_\lambda h_\mu\right\}$$

must be always negative or always positive for arbitrary values of $h_1, h_2, \cdots, h_n$.

II. THEORY OF THE HOMOGENEOUS QUADRATIC FORMS

53. The three kinds of quadratic forms, viz., *definite*, *semi-definite*, and *indefinite*, were defined in § 13.

As we have already indicated in § 13, it is seen that if the homogeneous function is an *indefinite* form, the function $f(x_1, x_2, \cdots, x_n)$ has neither a maximum nor a minimum upon the position $(a_1, a_2, \cdots, a_n)$; for if the right-hand member of [7] is positive, say, for a definite system of values of the h's, then in accordance with the definition of the indefinite quadratic forms we can find in the immediate neighborhood of the first system a second system of values of the h's for which the right-hand side of the equation [7] is negative; consequently, also, the difference

$$f(x_1, x_2, \cdots, x_n) - f(a_1, a_2, \cdots, a_n)$$

is negative, so that therefore no maximum or minimum is permissible for the position $(a_1, a_2, \cdots, a_n)$.

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If, then, the second derivatives of the function $f(x_1, x_2, \cdots, x_n)$ do not all vanish at the position $(a_1, a_2, \cdots, a_n)$, it follows, besides the equations [6], as a further condition for the existence of an extreme of the function $f(x_1, x_2, \cdots, x_2)$ that the terms of the second dimension in [4] must be a *definite* quadratic form, if we exclude what we have called the semi-definite case.

The question next arises: Under what conditions is in general a homogeneous quadratic form

$$[8] \qquad \phi(x_1, x_2, \cdots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\lambda x_\mu, \qquad A_{\lambda\mu} = A_{\mu\lambda}$$

a *definite* quadratic form?

54. Before we endeavor to answer this question we must yet consider some known properties of the homogeneous functions of the second degree.

Suppose that in the function $\phi(x_1, x_2, \cdots, x_n)$, in the place of $(x_1, x_2, \cdots, x_n)$, homogeneous linear functions of these quantities

$$[9] \qquad y_\lambda = \sum_{\mu=1}^{\mu=n} c_{\lambda\mu} x_\mu \quad (\lambda = 1, 2, \cdots, n)$$

are substituted, which are subjected to the condition that inversely the x's may be linearly expressed in terms of the y's, and consequently the determinant

$$[10] \qquad \begin{vmatrix} c_{11}, c_{12}, \cdots, c_{1n} \\ c_{21}, c_{22}, \cdots, c_{2n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_{n1}, c_{n2}, \cdots, c_{nn} \end{vmatrix} = \sum \pm c_{11} c_{22} \cdots c_{nn} \gtrless 0.$$

The function $\phi(x_1, x_2, \cdots, x_n)$ is thereby transformed into

$$[11] \qquad \phi(x_1, x_2, \cdots, x_n) = \psi(y_1, y_2, \cdots, y_n).$$

Owing to this substitution it may happen that $\psi(y_1, y_2, \cdots, y_n)$ does not contain one of the variables y, so that $\phi(x_1, x_2, \cdots, x_n)$ is expressible as a function of less than n variables.

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To find the condition for this write

$$[12] \qquad \phi_\lambda = \frac{1}{2}\frac{\partial \phi}{\partial x_\lambda} = \sum_\mu A_{\lambda\mu} x_\mu \quad (\lambda = 1, 2, \cdots, n).$$

If ψ is independent of one of the y's, say y_n, so that consequently $\frac{\partial \psi}{\partial y_n} = 0$, then from the n equations

$$[13] \qquad 2\phi_\lambda = \sum_{\mu=1}^{\mu=n}\left(\frac{\partial \psi}{\partial y_\mu}\frac{\partial y_\mu}{\partial x_\lambda}\right) = \sum_{\mu=1}^{\mu=n}\left(c_{\mu\lambda}\frac{\partial \psi}{\partial y_\mu}\right) \quad (\lambda = 1, 2, \cdots, n)$$

we may eliminate the $n-1$ unknown quantities $\frac{\partial \psi}{\partial y_1}, \frac{\partial \psi}{\partial y_2}, \cdots, \frac{\partial \psi}{\partial y_{n-1}}$. We thus have among the ϕ's an equation of the form

$$[14] \qquad \sum_{\mu=1}^{\mu=n} k_\mu \phi_\mu = 0,$$

where the k's are constants.

Owing to equations [12] this means that the determinant of the given quadratic form vanishes, that is,

$$[15] \qquad \sum \pm A_{11} A_{22} \cdots A_{nn} = 0.$$

We note here the following formulas:

$$[16] \qquad \begin{cases} \sum_\lambda \{\phi_\lambda(x_1, x_2, \cdots, x_n) x'_\lambda\} = \sum_{\lambda,\mu} A_{\lambda\mu} x_\mu x'_\lambda, \\ \sum_\lambda \{\phi_\lambda(x'_1, x'_2, \cdots, x'_n) x_\lambda\} = \sum_{\lambda,\mu} A_{\lambda\mu} x'_\mu x_\lambda, \\ \qquad\qquad = \sum A_{\mu\lambda} x'_\lambda x_\mu = \sum A_{\lambda\mu} x'_\lambda x_\mu; \end{cases}$$

and consequently

$$[17] \qquad \sum \{\phi_\lambda(x_1, x_2, \cdots, x_n) x'_\lambda\} = \sum \{\phi_\lambda(x'_1, x'_2, \cdots, x'_n) x_\lambda\}.$$

There exists, further, the well-known Euler's theorem for homogeneous functions:

$$[18] \qquad \sum_\lambda \{\phi_\lambda(x_1, x_2, \cdots, x_n) x_\lambda\} = \phi(x_1, x_2, \cdots, x_n).$$

It is also easy to show reciprocally that if, as above, the equation [15] is true, the function ϕ consists of less than n variables.

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For if we assume that equation [15], or, what amounts to the same thing, an identical relation of the form [14] exists, and if we substitute in $\phi(x_1, x_2, \cdots, x_n)$ the quantities $x_\lambda + tk_\lambda$ in the place of $x_\lambda (\lambda = 1, 2, \cdots, n)$ and develop with respect to powers of t, we then have

$$\phi(x_1 + tk_1, x_2 + tk_2, \cdots, x_n + tk_n) = \phi(x_1, x_2, \cdots, x_n) + 2t \sum \{k_\lambda \phi_\lambda (x_1, x_2, \cdots, x_n)\} + t^2 \phi(k_1, k_2, \cdots, k_n).$$

It follows, when we take into consideration the equations [14] and [18], since the equation [14] is true for every system of values $(x_1, x_2, \cdots, x_n)$, that

$$\phi(x_1 + tk_1, \cdots, x_n + tk_n) = \phi(x_1, \cdots, x_n).$$

Hence, if the equation [15] exists, or if the k's satisfy the equation [14] for every system of values $(x_1, x_2, \cdots, x_n)$, then $\phi(x_1, x_2, \cdots, x_n)$ remains invariantive if $x_\lambda + tk_\lambda$ is written for x_λ, where t is an arbitrary quantity.

Consequently, it being presupposed that $k_\nu \leqq 0$, if t is so determined that the argument $x_\nu + tk_\nu = 0$, we have

$$[19] \quad \phi(x_1, x_2, \cdots, x_n) = \phi\left(x_1 - \frac{k_1}{k_\nu} x_\nu, x_2 - \frac{k_2}{k_\nu} x_\nu, \cdots, x_{\nu-1} - \frac{k_{\nu-1}}{k_\nu} x_\nu, 0, x_{\nu+1} - \frac{k_{\nu+1}}{k_\nu} x_\nu, \cdots, x_n - \frac{k_n}{k_\nu} x_\nu\right),$$

where ϕ is expressed as a function of less than n variables. We therefore have the theorem

The vanishing of the determinant $\sum \pm A_{11} A_{22} \cdots A_{nn}$ *is the necessary and sufficient condition that a homogeneous quadratic function* $\phi(x_1, x_2, \cdots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\lambda x_\mu$ *be expressible as a function of* $n - 1$ *variables.*

55. We return to the question proposed at the end of § 53, and to have a definite case before us assume that the problem is: *Determine the condition under which the function* $\phi(x_1, x_2, \cdots, x_n)$ *is invariably positive.* The second case where $\phi(x_1, x_2, \cdots, x_n)$ is to be invariably negative is had at once if $-\phi$ is written in the place of ϕ.

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We shall first show, following a method due to Weierstrass,* that *every homogeneous function of the second degree* $\phi(x_1, x_2, \cdots, x_n)$ *may be expressed as an aggregate of squares of linear functions of the variables.*

56. In the proof of the above theorem it is assumed that $\phi(x_1, x_2, \cdots, x_n)$ cannot be expressed as a function of $n-1$ variables; it follows, therefore, that the inequality

$$[20] \qquad \sum \pm A_{11}A_{22}\cdots A_{nn} \gtrless 0$$

is true and that therefore it is not possible to determine constants k, so that the equation $\sum_{i=1}^{i=n} k_i\phi_i = 0$ exists identically.

If, then, y is a linear function of the x's having the form

$$[21] \qquad y = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

and if g is a certain constant, then the expression $\phi - gy^2 (= \overline{\phi}$, say), after the theorem proved above, can be expressed as a function of only $n-1$ variables if the constants $k_1, k_2, \cdots, k_n$ may be so determined that

$$\sum_{\lambda=1}^{\lambda=n} k_\lambda \overline{\phi}_\lambda = \sum_{\lambda=1}^{\lambda=n} \{k_\lambda(\phi_\lambda - gyc_\lambda)\} = 0$$

or

$$[22] \qquad \sum_{\lambda=1}^{\lambda=n} k_\lambda\phi_\lambda - gy\sum_{\lambda=1}^{\lambda=n} k_\lambda c_\lambda = 0.$$

From the assumption made regarding [20] it follows, on the one hand, that the inequality

$$[23] \qquad \sum_\lambda k_\lambda c_\lambda \gtrless 0$$

must exist. This is the only restriction placed upon the c's. On the other hand, in virtue of the n linear equations

$$[24] \qquad \sum_\mu A_{\lambda\mu}x_\mu = \phi_\lambda \quad (\lambda = 1, 2, \cdots, n)$$

* See also Lagrange, *Misc. Taur.*, Vol. I (1759), p. 18, and *Mécanique*, Vol. I, p. 3; Gauss, *Disq. Arithm.*, p. 271; *Theoria Comb. Observ.*, p. 31, etc.

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the quantities $x_1, x_2, \cdots, x_n$ may be expressed as linear functions of $\phi_1, \phi_2, \cdots, \phi_n$, and, consequently, by the substitution of these values of $x_1, x_2, \cdots, x_n$ in [21] y takes the form

$$[25] \qquad y = \sum_{\nu=1}^{\nu=n} e_\nu \phi_\nu,$$

where the e_ν are constants, which are composed of the constants $A_{\mu\lambda}$ and c_λ.

But from equation [22] it follows that

$$y = \frac{1}{g} \frac{\sum k_\lambda \phi_\lambda}{\sum k_\lambda c_\lambda}.$$

Such a representation of the ϕ_λ, however, since we have to do with linear equations, can be effected only in one way.

We therefore have $y = \dfrac{1}{g} \dfrac{\sum_{\mu=1}^{\mu=n} k_\mu \phi_\mu}{\sum_{\mu=1}^{\mu=n} k_\mu c_\mu} = \sum_{\lambda=1}^{\lambda=n} e_\lambda \phi_\lambda,$

from which it follows that

$$k_\lambda = g e_\lambda \sum_{\mu=1}^{\mu=n} k_\mu c_\mu \quad (\lambda = 1, 2, \cdots, n).$$

Through the substitution of these values in [22] it is seen that

$$\sum_{\lambda=1}^{\lambda=n} e_\lambda \phi_\lambda - g y \sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda = 0;$$

consequently, owing to the relation [25], we have

$$[26] \qquad \frac{1}{g} = \sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda \quad \text{or} \quad g = \frac{1}{\sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda}.$$

This value of g may be expressed in a different form; for from [25] and [17] it follows that

$$[26^a] \quad y = \sum_{\nu=1}^{\nu=n} e_\nu \phi_\nu(x_1, x_2, \cdots, x_n) = \sum_{\nu=1}^{\nu=n} x_\nu \phi_\nu(e_1, e_2, \cdots, e_n).$$

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Comparing this result with [21], we have

[27] $$c_\nu = \phi_\nu(e_1, e_2, \cdots, e_n) \quad (\nu = 1, 2, \cdots, n),$$

and consequently

[28] $$g = \frac{1}{\sum_{\lambda=1}^{\lambda=n} e_\lambda \phi_\lambda(e_1, e_2, \cdots, e_n)};$$

or, from [18], $$g = \frac{1}{\phi(e_1, e_2, \cdots, e_n)}.$$

Since the quantities $c_1, c_2, \cdots, c_n$ are perfectly arbitrary except the one restriction expressed by the inequality [23], the quantities $e_1, e_2, \cdots, e_n$ are, in consequence of the equation [27], completely arbitrary with the one limitation resulting from [28], viz., the function ϕ cannot vanish for the system of values $(e_1, e_2, \cdots, e_n)$; otherwise g would become infinite.

57. Reciprocally, if the quantities $e_1, e_2, \cdots, e_n$ are arbitrarily chosen, but with the restriction just mentioned, and if g is determined through [28], it may be proved that the expression $\bar{\phi}(= \phi - gy^2)$, where y has the form [25], may be expressed as a function of only $n-1$ variables. For, form the derivatives of this expression with respect to the different variables, and multiply each of the resulting quantities by the constants $e_1, e_2, \cdots, e_n$. Adding these products and noting [26^a] and [28], we have

$$\sum e_\lambda \phi_\lambda - gy \sum e_\lambda \phi_\lambda(e_1, e_2, \cdots, e_n) = \sum e_\lambda \phi_\lambda - y.$$

The expression on the right-hand side is zero from [25]. Hence n constants may be chosen in such a way that the sum of the products of these constants and the derivatives of the expression $\phi - gy^2$ is identically zero, and also $\bar{\phi}(e_1, \cdots, e_n) = 0$ (cf. [18]).

58. Substitute $x_\lambda + te_\lambda$ for $x_\lambda (\lambda = 1, 2, \cdots, n)$ in $\bar{\phi}$; if one of these arguments is made equal to zero, we have, as in § 54,

$$\phi(x_1, x_2, \cdots, x_n) - gy^2 = \bar{\phi}\left(x_1 - \frac{e_1}{e_k} x_k, \cdots, x_k - \frac{e_{k-1}}{e_k} x_k, 0, x_{k+1} - \frac{e_{k+1}}{e_k} x_k, \cdots, x_n - \frac{e_n}{e_k} x_k\right),$$

or, if the new arguments are represented by $x'_1, x'_2, \cdots, x'_{n-1}$,

$$\phi(x_1, x_2, \cdots, x_n) - gy^2 = \bar{\phi}(x'_1, x'_2, \cdots, x'_{n-1}).$$

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Employing the same method of procedure with $\overline{\phi}(x'_1, x'_2, \cdots, x'_{n-1})$ as was done with $\phi\ (x_1, x_2, \cdots, x_n)$, we come finally to the function of only one variable, which, being a homogeneous function of the second degree, is itself a square. Hence we have the given homogeneous function $\phi(x_1, x_2, \cdots, x_n)$ expressed as the sum of squares of linear homogeneous functions of the variables. If the coefficients of ϕ are real, as also the quantities e, the coefficients g are also real, and since the quantities e may with a single limitation be arbitrarily chosen, it follows that a transformation of such a kind that the result shall be a real one may be performed in an infinite number of ways.*

59. If, now, the expression

$$[29] \qquad \phi(x_1, x_2, \cdots, x_n) = g_1 y_1^2 + g_2 y_2^2 + \cdots + g_n y_n^2$$

is to be invariably positive for real values of the variables and equal to zero only when the variables themselves all vanish, then all the qualities $g_1, g_2, \cdots, g_n$ must be positive; for if this were not the case, but g_1, say, were negative, then, since the y's are, independently of one another, linear homogeneous functions of the x's, we could so choose the x's that all the y's except y_1 would vanish, and consequently, contrary to our assumption, $\phi(x_1, x_2, \cdots, x_n)$ would be negative. Furthermore, none of the g's can vanish; for if g_1, say, were zero, we might so choose a system of values $x_1, x_2, \cdots, x_n$, in which at least not all the quantities $x_1, x_2, \cdots, x_n$ were zero, that all the y's would vanish except y_1, and consequently ϕ could then be zero without the vanishing of all the variables $x_1, x_2, \cdots, x_n$.

Reciprocally, the condition of $g_1, g_2, \cdots, g_n$ being all positive is also sufficient for ϕ to be invariably positive for real values of the variables, and for ϕ to be equal to zero only when all the variables vanish.

60. In order to have, in as definite form as possible, the expression of ϕ as a sum of squares, we shall give to the expression [26] for g still a third form.

*See Burnside and Panton, *Theory of Equations* (1892), p. 430. In this connection it is of interest to note the *Theorem of Inertia* of Sylvester, *Coll. Math. Papers*, Vol. I, pp. 380, 511. See also Hermite, *Œuvres*, Vol. I, p. 429.

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In connection with [12] it follows from [27] that

$$c_\nu = \sum_{\mu=1}^{\mu=n} A_{\nu\mu} e_\mu \quad (\nu = 1, 2, \cdots, n).$$

Denote by A the determinant of these equations, which from [20] is not identically zero, that is,

[30] $$A = \sum \pm A_{11} A_{22} \cdots A_{nn}.$$

We have as the solution of the preceding equation

$$e_\mu = \frac{1}{A} \sum_{\lambda=1}^{\lambda=n} \frac{\partial A}{\partial A_{\lambda\mu}} c_\lambda \quad (\mu = 1, 2, \cdots, n).$$

It follows from this in connection with [26] that

[31] $$g = \frac{A}{\sum_{\lambda,\mu} \frac{\partial A}{\partial A_{\lambda\mu}} c_\lambda c_\mu},$$

an expression in which the c's are subject only to the one condition that

$$\sum \frac{\partial A}{\partial A_{\lambda\mu}} c_\lambda c_\mu$$

is *not* identically zero.

61. It is shown next that we may separate from $\phi(x_1, x_2, \cdots, x_n)$ the square of a single variable in such a way that the resulting function contains only $n-1$ variables.

For example, in order that the expression $\phi - gx_n^2$ be expressed as a function of $n-1$ variables, we may choose for g the value [31], after we have written in this expression $c_\lambda = 0 (\lambda = 1, 2, \cdots, n-1)$, while to c_n is given the value *unity*.

From this it is seen that

$$g = \frac{A}{\frac{\partial A}{\partial A_{nn}}} = \frac{A}{A_1},$$

where A_1 is the determinant of the quadratic form $\phi(x_1, x_2, \cdots, x_{n-1}, 0)$. Of course this determinant must be different from zero.

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Hence we may write

$$\phi(x_1, x_2, \cdots, x_n) = \frac{A}{A_1} x_n^2 + \bar{\phi}(x_1', x_2', \cdots, x_{n-1}'),$$

where

$$\bar{\phi}(x_1', x_2', \cdots, x_{n-1}') = \phi\left(x_1 - \frac{e_1}{e_n} x_n, x_2 - \frac{e_2}{e_n} x_n, \cdots, x_{n-1} - \frac{e_{n-1}}{e_n} x_n, 0\right).$$

We may then proceed with $\bar{\phi}$ just as has been done with ϕ by separating the square of x_{n-1}', etc.

After the separation of μ squares from the original function ϕ, we notice that the determinant of the resulting function in $n - \mu$ variables is the same as the determinant of the function which results from the original function ϕ when we cause the μ last variables in it to vanish. If this determinant is denoted by A_μ, we have the following expression for ϕ:

$$\phi(x_1, x_2, \cdots, x_n)$$
$$= \frac{A}{A_1} x_n^2 + \frac{A_1}{A_2} x_{n-1}'^2 + \cdots + \frac{A_{n-2}}{A_{n-1}} (x_2^{(n-2)})^2 + A_{n-1} (x_1^{(n-1)})^2.$$

62. If now ϕ is to be invariably positive and equal to zero only when all the variables vanish, the coefficients on the right-hand side of the above expression must all be greater than zero. We therefore have the theorem

In order that the quadratic form

$$\phi(x_1, x_2, \cdots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\mu x_\lambda, \quad A_{\lambda\mu} = A_{\mu\lambda},$$

be a definite form and remain invariably positive, it is necessary and sufficient that the quantities $A_1, A_2, \cdots, A_{n-1}$, *which are defined through the equation* $A_\mu = \sum \pm A_{11} A_{22} \cdots A_{n-\mu, n-\mu}$, *be all positive and different from zero. If, on the other hand, the quadratic form is to remain invariably negative, then of the quantities* $A_{n-1}, A_{n-2}, \cdots, A_1, A$, *the first must be negative, and the following must be alternately positive and negative* (see Stolz, *Wiener Bericht*, Vol. LVIII (1868), p. 1069).

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III. APPLICATION OF THE THEORY OF QUADRATIC FORMS TO THE PROBLEM OF MAXIMA AND MINIMA STATED IN §§ 47–51

63. By establishing the criterion of the previous section the original investigation regarding the maxima and minima of the function $f(x_1, x_2, \cdots, x_n)$ is finished. The result established in § 57 may in accordance with the definitions given in § 52 be expressed as follows: *In order that an extreme of the function* $f(x_1, x_2, \cdots, x_n)$ *may in reality enter on the position* $(a_1, a_2, \cdots, a_n)$ *which is determined through the equations* [6], *it is sufficient, if the second derivatives of the function do not all vanish at this position, that the aggregate of the terms of the second degree of the equation* [4] *be a* ***definite*** * *quadratic form; if, however, the form vanishes for other values of the variables without changing sign* (*that is, is* ***semi-definite***), *then a determination as to whether an extreme in reality exists is not effected in the manner indicated and requires further investigation, as is seen below.*

In virtue of the theorem stated in § 53, an extreme will enter for a system of real values of the equation [6] if the homogeneous function of the second degree

$$\sum_{\lambda,\mu}\left\{\left(\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x_\lambda \partial x_\mu}\right)_{a_1, a_2, \cdots, a_n} h_\lambda h_\mu\right\},$$

that is, if

$$\sum_{\lambda,\mu}\{f_{\lambda,\mu}(a_1, a_2, \cdots, a_n) h_\lambda h_\mu\},$$

is a definite quadratic form; in other words (§ 62), *there will be a minimum on the position* $(a_1, a_2, \cdots, a_n)$ *if the quotients*

$$\frac{F_0}{F_1}, \quad \frac{F_1}{F_2}, \quad \cdots, \quad \frac{F_{n-2}}{F_{n-1}},$$

where $F_\mu = \sum \pm f_{11} f_{22} + \cdots f_{n-\mu, n-\mu}$, *are all positive, a maximum if they are all negative. In both cases the quotients must be different from zero.*

* Lagrange, *Théorie des Fonctions*, pp. 283, 286; see also Cauchy, *Calc. différ.*, p. 234.

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This last condition is only another form of what was said above, viz., that $\sum_{\lambda,\mu} f_{\lambda\mu}h_\lambda h_\mu$ must *not* be a semi-definite form. For if, say,

$$F_0 = \sum \pm f_{11}f_{22}\cdots f_{nn} = 0,$$

then the summation $\sum_{\lambda,\mu} f_{\lambda\mu}h_\lambda h_\mu$ being denoted by $\phi(h_1, h_2, \cdots, h_n)$, this equation would directly imply the existence of a relation of the form

$$\sum_\nu k_\nu \phi_\nu(h_1, h_2, \cdots, h_n) = 0,$$

where the k_ν are constants which do not all simultaneously vanish.

If, therefore, k_n, say, is different from zero, we may write

$$\phi_n = -\frac{1}{k_n}\sum_{\nu=1}^{\nu=n-1} k_\nu \phi_\nu,$$

and with the help of this relation we have from the equation

$$\phi(h_1, h_2, \cdots, h_n) = \sum_{\lambda=1}^{\lambda=n} \phi_\lambda(h_1, h_2, \cdots, h_n)h_\lambda = \phi_n h_n + \sum_{\lambda=1}^{\lambda=n-1} \phi_\lambda h_\lambda$$

the following relation

$$\phi(h_1, h_2, \cdots, h_n) = \sum_{\lambda=1}^{\lambda=n-1} \phi_\lambda(h_1, h_2, \cdots, h_n)\left(h_\lambda - \frac{k_\lambda}{k_n}h_n\right).$$

Now in this expression the arbitrary quantities h may be so chosen that

$$h_\lambda = \frac{k_\lambda}{k_n}h_n \quad (\lambda = 1, 2, \cdots, n),$$

and consequently the function $\phi(h_1, h_2, \cdots, h_n)$ would vanish without all the h's becoming simultaneously zero. This case we *cannot* treat in its generality.

Neglecting this case, it is seen that the problem of this chapter is completely treated; however, the conditions that a quadratic form shall be a definite one appear in a less symmetric form than we wish. It is due to the fact that we have given special preponderance to certain variables over the others.

We shall consequently take up the same subject again in the next chapter.

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64. The question is often regarding the *greatest* and the *least* values (the upper and lower limits) which a function may take when its variables vary in a given finite or infinite region. If this value corresponds to a system of values *within* the given region, then for this system the function will also be a maximum or a minimum in the sense derived above.

For example, let it be required to distribute a positive number a into $n+1$ summands, so that the product of the α_1th power of the first, the α_2th power of the second, etc., and finally the α_{n+1} power of the last summand will be a maximum.*

The quantities $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ are to be positive numbers. Let $x_1, x_2, \cdots, x_n, a - x_1 - x_2 - \cdots - x_n$ be the summands in question and write

$$U = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} (a - x_1 - x_2 - \cdots - x_n)^{\alpha_{n+1}}.$$

We must then determine when U or, what is the same thing, its natural logarithm, has its greatest value.

If we put the partial derivatives of $\log U$ equal to zero, we will have

$$\begin{aligned}
\frac{\partial \log U}{\partial x_1} &= \frac{\alpha_1}{x_1} - \frac{\alpha_{n+1}}{a - x_1 - \cdots - x_n} = 0, \\
\frac{\partial \log U}{\partial x_2} &= \frac{\alpha_2}{x_2} - \frac{\alpha_{n+1}}{a - x_1 - \cdots - x_n} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial \log U}{\partial x_n} &= \frac{\alpha_n}{x_n} - \frac{\alpha_{n+1}}{a - x_1 - \cdots - x_n} = 0.
\end{aligned}$$

These equations may be written

$$\frac{x_1}{\alpha_1} = \frac{x_2}{\alpha_2} = \cdots = \frac{x_n}{\alpha_n} = \frac{a - x_1 - \cdots - x_n}{\alpha_{n+1}} = \frac{a}{\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}},$$

the last term being had through addition of the preceding proportions.

If we call $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$ the values of the variables which satisfy these equations, we have

$$x_1^{(0)} = a \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_{n+1}}; \quad \cdots; \quad x_n^{(0)} = a \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_{n+1}}.$$

* Peano, § 137.

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The corresponding value of U is

$$U^{(0)} = a^{\alpha_1 + \cdots + \alpha_{n+1}} \frac{\alpha_1^{\alpha_1} \cdots \alpha_{n+1}^{\alpha_{n+1}}}{(\alpha_1 + \cdots + \alpha_{n+1})^{\alpha_1 + \cdots + \alpha_{n+1}}}.$$

To recognize whether U_0 is the greatest of the values of U, we may show that U is in fact a maximum for the system of values $x_1^{(0)}, \cdots, x_n^{(0)}$ and that this position lies on the interior of the realm of variability under consideration. For, let $x_1, x_2, \cdots, x_n$ be another system of positive values of the variables, for which also $a - x_1 - \cdots - x_n$ is positive, and substitute for the variables in $\log U$ the values

$$x_1^{(0)} + u(x_1 - x_1^{(0)}), \cdots, x_n^{(0)} + u(x_n - x_n^{(0)}), \quad \text{where} \quad 0 < u < 1.$$

Since the partial derivatives of the first and second order of $\log U$ are continuous for all these systems of values, we have through the Taylor development, observing that the first derivatives vanish on the position $x_1^{(0)}, \cdots, x_n^{(0)}$,

$$\log U = \log U_0 - \frac{1}{2}\left[\frac{\alpha_1 (x_1^{(1)} - x_1^{(0)})^2}{(x_1^{(1)})^2} + \cdots + \frac{\alpha_n (x_n^{(1)} - x_n^{(0)})^2}{(x_n^{(1)})^2} + \frac{\alpha_{n+1}(x_1^{(1)} + \cdots + x_n^{(1)} - x_1^{(0)} - \cdots - x_n^{(0)})^2}{(a - x_1^{(1)} - \cdots - x_n^{(1)})^2}\right],$$

where $x_1^{(1)}, \cdots, x_n^{(1)}$ are values of the variables of the form

$$x_1^{(0)} + \theta(x_1 - x_1^{(0)}), \cdots, x_n^{(0)} + \theta(x_n - x_n^{(0)}), \quad \text{where} \quad 0 < \theta < 1.$$

The expression within the brackets is positive and different from zero, since it is assumed that the system of values $x_1, x_2, \cdots, x_n$ do not coincide with $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$. It follows that $\log U < \log U_0$ or $U < U_0$, so that U_0 is, in fact, the greatest value which U can assume.

We note that U takes a *smallest* value, viz., zero, if one of the summands into which a is distributed, vanishes. If we allow the summands to take negative values, it no longer follows that U_0 is the greatest of the values U.

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CHAPTER VI

THEORY OF MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES THAT ARE SUBJECTED TO SUBSIDIARY CONDITIONS. RELATIVE MAXIMA AND MINIMA

65. In the preceding investigations the variables $x_1, x_2, \cdots, x_n$ were completely independent of one another.

We now propose the problem: *Among all systems of values* $(x_1, x_2, \cdots, x_n)$ *find those which cause the function* $F(x_1, x_2, \cdots, x_n)$ *to have maximum and minimum values and which at the same time satisfy the equations of conditions:*

$$[1] \quad f_\lambda(x_1, x_2, \cdots, x_n) = 0 \quad (\lambda = 1, 2, \cdots, m;\ m < n),$$

where $f_\lambda(x_1, x_2, \cdots, x_n)$ *and* $F(x_1, \cdots, x_n)$ *are functions of the same character as* $f(x_1, x_2, \cdots, x_n)$ *in § 47.*

66. The natural way to solve the problem is to express by means of equations [1] m of the variables in terms of the remaining $n - m$ variables and write their values in $F(x_1, x_2, \cdots, x_n)$. This function would then depend only upon the $n - m$ variables which are independent of one another, and so the present problem would be reduced to the one of the preceding chapter.

In general, this method of procedure cannot be readily performed, since it is not always possible by means of equations [1] to represent in reality m variables as functions of the $n - m$ remaining variables. A more practicable method must therefore be sought.

67. If $(a_1, a_2, \cdots, a_n)$ is any system of values of the quantities $x_1, x_2, \cdots, x_n$ which satisfy the equations [1], then of the systems of values

$$(x_1 = a_1 + h_1,\ x_2 = a_2 + h_2,\ \cdots,\ x_n = a_n + h_n),$$

in the neighborhood of $(a_1, \cdots, a_n)$, only those which satisfy the equations [1] may be considered; that is, we must have

$$[2] \quad f_\lambda(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) = 0 \quad (\lambda = 1, 2, \cdots, m).$$

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Hence by Taylor's theorem the h's satisfy the equations

$$[3] \quad \sum_{\mu=1}^{\mu=n} \{f_{\lambda\mu}(a_1, a_2, \cdots, a_n)h_\mu\} + [h_1, h_2, \cdots, h_n]_\lambda^2 = 0 \qquad (\lambda = 1, 2, \cdots, m),$$

where $[h_1, h_2, \cdots, h_n]_\lambda^2$ denotes the terms of the second and higher dimensions in the respective variables.

68. It being assumed that at least one of the determinants of the mth order which can be produced by neglecting $n - m$ columns from the system of $m \cdot n$ quantities

$$[4] \quad \begin{cases} f_{11}, f_{12}, \cdots, f_{1n}, \\ f_{21}, f_{22}, \cdots, f_{2n}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_{m1}, f_{m2}, \cdots, f_{mn}, \end{cases}$$

is different from zero, then (see §§ 135 and 136) m of the quantities h may be expressed through the remaining $n - m$ quantities (which may be denoted by $k_1, k_2, \cdots, k_{n-m}$) in the form of power series as follows:

$$[5] \quad h_\lambda = (k_1, k_2, \cdots, k_{n-m})_\lambda^{(1)} + (k_1, k_2, \cdots, k_{n-m})_\lambda^{(2)} + \cdots \qquad (\lambda = 1, 2, \cdots, m),$$

where the upper indices denote the dimensions of the terms with which they are associated. These series converge in the manner indicated in § 136; they satisfy identically the equations [2] and furnish, if the quantities $k_1, k_2, \cdots, k_{n-m}$ are taken sufficiently small, all values of the m quantities h which satisfy these equations.

69. The condition that one of the determinants in the preceding article be different from zero is in general satisfied; there are, however, special cases where this is not the case. A geometrical interpretation will explain these exceptions.

Let F and an equation of condition $f = 0$ contain only three variables x_1, x_2, and x_3.

The equation of condition $f(x_1, x_2, x_3) = 0$ represents then a surface upon which the point (x_1, x_2, x_3) is to lie and for which $F(x_1, x_2, x_3)$ is to have a maximum or minimum value.

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The determinants of the first order in the development

$$f(a_1+h_1,\ a_2+h_2,\ a_3+h_3)-f(a_1,\ a_2,\ a_3)$$

with respect to powers of h_1, h_2, and h_3 cannot all be equal to zero; that is, all the terms of the first dimension cannot vanish, the single terms being these determinants; and this means that the surface $f=0$ cannot have a singularity at the point in question.

Take next two equations of condition $f_1=0$ and $f_2=0$ between three variables x_1, x_2, and x_3. Considered together they represent a curve, and the condition that the corresponding determinants of the second order cannot all be zero means here that the curve at the point in question cannot have a singularity.

70. If the values of the m quantities h_λ are substituted in the difference

$$F(x_1,\ x_2,\ \cdots,\ x_n)-F(a_1,\ a_2,\ \cdots,\ a_n),$$

this expression then depends only upon the $n-m$ variables $k_1,\ k_2,\ \cdots,\ k_{n-m}$, that are independent of one another and may consequently for sufficiently small values of these variables be developed in the form

$$[6] \qquad \begin{aligned} &F(x_1,\ x_2,\ \cdots,\ x_n)-F(a_1,\ a_2,\ \cdots,\ a_n)\\ &\quad=\sum_{\rho=1}^{\rho=n-m} C_\rho k_\rho+\frac{1}{2}\sum_{\rho,\ \sigma} C_{\rho\sigma}k_\rho k_\sigma+\cdots. \end{aligned}$$

It was seen (§ 51) that, in order to have a maximum or minimum on the position $(a_1,\ a_2,\ \cdots,\ a_n)$, it is necessary that the terms of the first dimension vanish, and consequently

$$[7] \qquad C_\rho=0 \quad (\rho=1,\ 2,\ \cdots,\ n-m).$$

71. This condition may be easily expressed in another manner. We may obtain the quantities e if, in the development

$$\begin{aligned} &F(x_1,\ x_2,\ \cdots,\ x_n)-F(a_1,\ a_2,\ \cdots,\ a_n)\\ &\quad=\sum_{\mu=1}^{\mu=n} F_\mu h_\mu+\frac{1}{2}\sum_{\mu,\ \nu} F_{\mu\nu}h_\mu h_\nu+\cdots, \end{aligned}$$

we substitute in the terms of the first dimension the values of the m quantities from [5] and arrange the result according to the

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quantities $k_1, k_2, \cdots, k_{n-m}$. In other words, the equations [7] express the condition that $\sum_{\mu=1}^{\mu=n} F_\mu h_\mu$ must vanish identically for all systems of values of the h's that satisfy the m equations [3] after they have been reduced to their linear terms. These are the m equations

$$[8] \qquad \sum_{\mu=1}^{\mu=n} f_{\lambda\mu} h_\mu = 0 \quad (\lambda = 1, 2, \cdots, m).$$

Now multiplying these m equations* respectively by m arbitrary quantities $e_1, e_2, \cdots, e_m$, and adding the results to the equation

$$\sum_{\mu=1}^{\mu=n} F_\mu h_\mu = 0,$$

we have the following equation:

$$[9] \qquad \sum_{\mu=1}^{\mu=n} \{(F_\mu + e_1 f_{1\mu} + e_2 f_{2\mu} + \cdots + e_m f_{m\mu}) h_\mu\} = 0.$$

But the e's may be so determined that those terms in this summation drop out which contain the m quantities h, which are expressed in [5] through the $n-m$ other h's; by causing these terms to vanish, a system of m linear equations is obtained, whose determinant by hypothesis is different from zero.

Since the terms which remain of equation [9] are multiplied by the completely arbitrary quantities $k_1, k_2, \cdots, k_{n-m}$, it is not possible for this equation to exist unless each of the single coefficients is equal to zero.

Consequently we have as the first necessary condition for the appearance of a maximum or minimum the existence of the following system of n equations,

$$F_\mu + e_1 f_{1\mu} + e_2 f_{2\mu} + \cdots + e_m f_{m\mu} = 0 \quad (\mu = 1, 2, \cdots, n),$$

in the sense that if m of these equations exist independently of one another, the remaining $n-m$ of them must be identically satisfied through the substitution of the e's which are derived

*This method is due to Lagrange, *Théorie des Fonctions*, p. 268; see also Gauss (*Theoria Comb. Observ. Supp.* § 11).

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from the first m equation, it being of course presupposed that the system of values $(a_1, a_2, \cdots, a_n)$ has already been so chosen that the equations [1] are satisfied.

Taking everything into consideration we may say: *In order that the function* $F(x_1, x_2, \cdots, x_n)$ *have a maximum or minimum on any position* $(a_1, a_2, \cdots, a_n)$, *it is necessary that the* $n + m$ *equations*

$$[10] \qquad \begin{cases} \dfrac{\partial F}{\partial x_\mu} + e_1 \dfrac{\partial f_1}{\partial x_\mu} + e_2 \dfrac{\partial f_2}{\partial x_\mu} + \cdots + e_m \dfrac{\partial f_m}{\partial x_\mu} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mu = 1, 2, \cdots, n), \\ f_\lambda(x_1, x_2, \cdots, x_n) = 0 \quad (\lambda = 1, 2, \cdots, m) \end{cases}$$

be satisfied by a system of real values of the $n + m$ *quantities* $x_1, x_2, \cdots, x_n, e_1, e_2, \cdots, e_m$.

72. These deductions were made under the one assumption that at least one of the determinants of the mth order which can be formed out of the $m \cdot n$ quantities [4] through the omission of $n - m$ columns does not vanish. This condition was necessary both for the determination of the quantities h, which satisfy the equations [2], and also for the determination of the m factors $e_1, e_2, \cdots, e_m$.

It may happen* that a maximum or minimum of the function F enters on the position $(a_1, a_2, \cdots, a_n)$ even when the above condition is not satisfied. For if it is possible in any way to determine all systems of values of the h's not exceeding certain limits that satisfy the equations [2], the equations [7] together with the equations [1] are sufficient in number to determine the n quantities $a_1, a_2, \cdots, a_n$.

When the above assumption is not satisfied, the equations [8] exist identically, and consequently the equations [3], which serve to determine the h's, begin with terms of the second dimension. We may often in this case proceed advantageously by introducing in the place of the original variables a system of $n - m$ new variables so chosen that when they are substituted in the given equations of condition they identically satisfy them.

* See Stolz, p. 257.

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73. To make clear what has been said, the following example will be of service; its general solution is given in the sequel (§ 91). *Find the shortest line which can be drawn from a given point to a given surface.* Upon the surface there are certain points of such a nature that the lines joining these points with the given point have the desired property and, besides, stand normal to the surface at these points.

If by chance it happens that one of these points is a double point (node) of the surface, so that at it we have $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, then in reality for this point the terms of the first dimension in the equations [2] drop out and we have the case just mentioned.

If the surface is the right cone

$$f(x, y, z) = 0 = x^2 + y^2 - z^2,$$

we may write

$$[11] \qquad \begin{cases} x = 2uv, \\ y = u^2 - v^2, \\ z = u^2 + v^2. \end{cases}$$

The equation of the surface is identically satisfied, and it is easily seen that we may express the quantities h_1, h_2, h_3 through two quantities k_1 and k_2 independent of each other even in the case where the required point of the surface is the vertex of the cone, that is, the point $x = 0 = y = z$, or $u = 0 = v$; and in fact in such a way that not only indefinitely small values of h_1, h_2, h_3 correspond to indefinitely small values of k_1, k_2 but also that all systems of values h_1, h_2, h_3 are had which satisfy the equation

$$f(x + h_1, y + h_2, z + h_3) = 0.$$

The variables, however, must be given at one time real, at another time purely imaginary, values if the equations [11] are to represent the entire surface of the cone; but in this manner the unavoidable trouble has taken such a direction that the proposed problem falls into two similar parts, which may be treated in full after the methods of Chapter V. In other cases we may proceed in a like manner. The special problem will each time of itself offer the most propitious method of procedure.

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74. We must now establish the criteria from which one can determine whether a maximum or minimum of $F(x_1, x_2, \cdots, x_n)$ really enters or not on a definite position $(a_1, a_2, \cdots, a_n)$, which has been determined in § 71 above.

One might consider this superfluous, since in virtue of the criteria given in the previous chapter a maximum or minimum will certainly enter if the aggregate of terms of the second dimension in [6] is a definite quadratic form of the nature indicated.

It is, however, desirable to determine the existence of a maximum or minimum without having previously made the development of the function in the form [6]; for in order to obtain the coefficients $C_{\rho\sigma}$ we must pay attention not only to the terms of the first dimension but also to the terms of the second dimension, when the values of [5] are substituted in the development of

$$F(x_1, x_2, \cdots, x_n) - F(a_1, a_2, \cdots, a_n) = \sum_{\mu=1}^{\mu=n} F_\mu h_\mu + \frac{1}{2} \sum_{\mu,\nu} F_{\mu\nu} h_\mu h_\nu + \cdots.$$

75. The above difficulty may be avoided if we multiply by the quantities $e_\mu (\mu = 1, 2, \cdots m)$ respectively each of the expressions [2] which vanish identically, add them thus multiplied to the above difference, and then develop the whole expression with respect to the powers of h.

Owing to equation [9] terms of the first dimension can no longer appear in this development, and we have, if we write

$$[12] \qquad F + \sum_{\mu=1}^{\mu=m} e_\mu f_\mu = G,$$

$$[13] \quad F(x_1, x_2, \cdots, x_n) - F(a_1, a_2, \cdots, a_n) = G(x_1, x_2, \cdots, x_n) - G(a_1, a_2, \cdots, a_n) = \frac{1}{2} \sum_{\mu,\nu} G_{\mu\nu} h_\mu h_\nu + \cdots.$$

We have, accordingly, the homogeneous function of the second degree $\sum_{\rho,\sigma} C_{\rho\sigma} k_\rho k_\sigma$ of the formula [6] if we substitute in $\sum_{\mu,\nu} G_{\mu\nu} h_\mu h_\nu$ the values [5] and consider only the terms of the first dimension

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in the process. If then the criteria of the preceding chapter are applied we can determine whether the function F possesses or not a maximum or minimum on the position $(a_1, a_2, \cdots, a_n)$.

76. The definite conditions that have been thus derived are unsymmetric for a twofold reason: on the one hand because in the determination of the quantities h some of them have been given preference over the others, and on the other hand because those expressions by means of which it is to be decided whether the function of the second degree is continuously positive or continuously negative have been formed in an unsymmetric manner from the coefficients of the function.

It is therefore interesting to derive a criterion which is free from these faults and which also indicates in many cases how the results will turn out. With this in view let us return to the problem already treated in the preceding chapter and propose the following more general theorem in quadratic forms.

I. THEORY OF HOMOGENEOUS QUADRATIC FORMS

77. THEOREM. *We have given a homogeneous function of the second degree*

$$[14] \qquad \phi(x_1, x_2, \cdots, x_n) = \sum_{\lambda,\mu} A_{\lambda\mu} x_\lambda x_\mu \quad (A_{\lambda\mu} = A_{\mu\lambda})$$

in n variables, which are subjected to the linear homogeneous equations of condition

$$[15] \qquad \theta_\lambda = \sum_{\mu=1}^{\mu=n} a_{\lambda\mu} x_\mu = 0 \quad (\lambda = 1, 2, \cdots, m;\ m < n);$$

we are required to find the conditions under which ϕ is invariably positive or invariably negative for all those systems of values of the variables which satisfy equations [15].

It is in every respect sufficient to solve this theorem with the limitation that the quantities x are subjected to the further condition

$$[16] \qquad x_1^2 + x_2^2 + \cdots + x_n^2 = 1;$$

for if $x_1^2 + x_2^2 + \cdots + x_n^2 = \rho^2$, then $\left(\frac{x_1}{\rho}\right)^2 + \left(\frac{x_2}{\rho}\right)^2 + \cdots + \left(\frac{x_n}{\rho}\right)^2 = 1.$

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Furthermore, if $x_1, \cdots, x_n$ satisfy [15], then, also, $\frac{x_1}{\rho}, \cdots, \frac{x_n}{\rho}$ satisfy these equations, while, since $\phi\left(\frac{x_1}{\rho}, \cdots, \frac{x_n}{\rho}\right) = \frac{1}{\rho^2}\phi(x_1, \cdots, x_n)$, the signs of the two quadratic forms are the same.

It is, therefore, in every respect admissible to add the equation [16]. We have, however, thereby gained an essential advantage: for owing to the condition [16] none of the variables can lie without the interval $-1 \cdots +1$; furthermore, since the function varies in a continuous manner, it must necessarily have an upper and a lower limit for these values of the variables $x_1, x_2, \cdots, x_n$; that is, among all systems of values which satisfy the equations [15] and [16] there must necessarily be one * which gives an upper limit and one which gives a lower limit of ϕ (see § 8).

We limit ourselves to the determination of the latter. By trial we can easily determine whether ϕ reaches its lower limit on the boundaries, that is, when one of the x's $= \pm 1$, while the others are all zero. If this lower limit is *not* reached *on* the boundaries, then ϕ has a minimum value *within* the boundaries (cf. § 64).

78. Through the addition of equation [16] the theorem of the preceding article is reduced to a problem in the theory of maxima and minima; for if the minimum value of $\phi(x_1, x_2, \cdots, x_n)$ is positive, ϕ is certainly a definite positive form.

Consequently, if we write

$$[17] \qquad G = \phi - e\left(\sum_{\lambda=1}^{\lambda=n} x_\lambda^2 - 1\right) + 2\sum_{\rho=1}^{\rho=m} e_\rho \theta_\rho,$$

then, in order to find the position at which there is a minimum value of the function, we have to form the system of equations

$$\frac{\partial G}{\partial x_\lambda} = 0 \quad (\lambda = 1, 2, \cdots, n).$$

This gives $$\frac{\partial \phi}{\partial x_\lambda} - 2ex_\lambda + 2\sum_{\rho=1}^{\rho=m} e_\rho \frac{\partial \theta_\rho}{\partial x_\lambda} = 0 \quad (\lambda = 1, 2, \cdots, n),$$

or,

$$[18] \qquad \sum_{\mu=1}^{\mu=n} A_{\lambda\mu} x_\mu - ex_\lambda + \sum_{\rho=1}^{\rho=m} e_\rho a_{\rho\lambda} = 0 \quad (\lambda = 1, 2, \cdots, n).$$

*Crelle's *Journal*, Vol. LXXII, p. 141; see also Serret, *Calc. diff. et int.*, pp. 17 et seq.

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From the $n+m+1$ equations

$$[19]\quad \begin{cases} \sum_{\mu=1}^{\mu=n} A_{\lambda\mu}x_\mu - ex_\lambda + \sum_{\rho=1}^{\rho=m} e_\rho a_{\rho\lambda} = 0 \quad (\lambda = 1, 2, \cdots, n), \\ \theta_\rho = \sum_{\mu=1}^{\mu=n} a_{\rho\lambda}x_\mu = 0 \quad (\rho = 1, 2, \cdots, m), \\ \sum_{\lambda=1}^{\lambda=n} x_\lambda^2 = 1 \end{cases}$$

the $n+m+1$ quantities $x_1, x_2, \cdots, x_n, e_1, e_2, \cdots, e_m, e$ may be determined. Since we know a priori that a minimum value of the function ϕ in reality exists on *one* position, we are certain that this system of equations must determine at least one real system of values.

Consequently the first $n+m$ linear homogeneous equations of [19] are consistent with one another and may be solved with respect to the unknown quantities $x_1, x_2, \cdots, x_n, e_1, e_2, \cdots, e_m$; their determinant must therefore vanish, and we must have

$$[20]\quad \Delta e \equiv \begin{vmatrix} A_{11}-e, & A_{12}, & \cdots, & A_{1n}, & a_{11}, \cdots, & a_{m1} \\ A_{21}, & A_{22}-e, & \cdots, & A_{2n}, & a_{12}, \cdots, & a_{m2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1}, & A_{n2}, & \cdots, & A_{nn}-e, & a_{1n}, \cdots, & a_{mn} \\ a_{11}, & a_{12}, & \cdots, & a_{1n}, & 0, \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}, & a_{m2}, & \cdots, & a_{mn}, & 0, \cdots, & 0 \end{vmatrix} = 0.$$

The equation $\Delta e = 0$ is clearly of the $n-m$th degree in e. The minimum value of ϕ is necessarily contained among the roots of this equation; for if we multiply the equations [18] respectively by $x_1, x_2, \cdots, x_n$ and add the results, we have

$$[21]\qquad \phi(x_1, x_2, \cdots, x_n) = e,$$

it being presupposed that the system of values $(x_1, x_2, \cdots, x_n)$, together with the quantities $e_1, e_2, \cdots, e_m$, satisfies the system of equations [19], which is only possible if e is a root of the equation $\Delta e = 0$. Furthermore, among the systems of values x which satisfy the system of equations [19] that system is also to be

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found which calls for the minimum, and since the value of the function which belongs to such a system of values is always a root of equation [20], it follows also that the required minimal value of ϕ must be contained among the roots of this equation.

As already remarked, this minimal value must be positive if ϕ is to be continuously positive for the systems of values of the x's under consideration, and from this it follows that Δe must have only positive roots. For if one root of this equation was negative, then for this root we could determine a system of values $x_1, x_2, \cdots, x_n, e_1, e_2, \cdots, e_m$ for which, as seen from [21], ϕ is likewise negative.

Hence, in order that ϕ be continuously positive for all systems of values of the x's which satisfy the equations [15], *it is necessary and sufficient that the equation $\Delta e = 0$ have only positive roots.**

The question next arises, When does the equation $\Delta e = 0$ have only positive roots? It may be answered in a completely rigorous manner by means of Sturm's theorem;† but the investigation is somewhat difficult; and the symmetry, which we especially wish to preserve, would be lost when we applied Sturm's theorem.

For develop the determinant according to powers of e as follows:

$$[22] \quad e^{n-m} - B_1 e^{n-m-1} + B_2 e^{n-m-2} - \cdots + (-1)^{n-m} B_{n-m} = 0;$$

then if all the roots of this equation are real and positive, the coefficients B must be all positive, and, reciprocally, if the roots of this equation are real and the B's are all greater than 0, the roots of the equation $\Delta e = 0$ are all positive. The form is then a definite quadratic form. The necessary and sufficient condition that the form be *not* a definite one is that $e = 0$ be the *smallest* root of the equation above.

* See Zajaczkowski, *Annals of the Scientific Society of Cracow*, Vol. XII (1867); see also Richelot, *Astronom. Nachr.*, Vol. XLVIII, p. 273.

† Burnside and Panton, *Theory of Equations*, chap. ix; Hermite, *Crelle*, Vol. LII, p. 43; Serret, *Algèbre Sup.*, Vol. I (1866), p. 581; Kronecker, *Berlin. Monatsbericht*, February, 1873.

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79. We shall first show that all the roots of the equation $\Delta e = 0$ are real for the case where no equations of conditions are present. (See J. Petzval, Haidinger's *Naturw. Abh. II* (1848), p. 115.)

Equation [20] reduces then to the form

$$[23] \qquad \begin{vmatrix} A_{11}-e, & A_{12}, & \cdots, & A_{1n} \\ A_{21}, & A_{22}-e, & \cdots, & A_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ A_{n1}, & A_{n2}, & \cdots, & A_{nn}-e \end{vmatrix} = 0, \text{ where } A_{\lambda\mu} = A_{\mu\lambda},$$

an equation which is called the *equation of secular variations* and plays an important rôle in many analytical investigations; for example, in the determination of the secular variations of the orbits of the planets, as well as in the determination of the principal axes of lines and surfaces of the second degree.*

80. Weierstrass's proof†, which is very simple, that all the roots of this equation are real, depends only upon the theorem that if the determinant of a system of n homogeneous equations vanishes, it is always possible to satisfy the equations through values of the unknown quantities that are not all equal to zero.

Instead of the equation [16] we subject the variables to the somewhat more general equation

$$\psi(x_1, x_2, \cdots, x_n) = 1,$$

where ψ denotes a homogeneous function of the second degree, which is always positive‡ and is only equal to 0 when the variables themselves vanish.

* In this connection the reader is referred to Laplace, *Mém. de Paris*, Vol. II (1772), pp. 293–363; Euler, *Mém. de Berlin* (1749–1750); *Theoria motus corp. sol.*, chap. v (1765); Lagrange, *Mém. de Berlin* (1773), p. 108; Poison et Hachette, *Journ. de l'École Polytechn.*, Cah. XI (1802), p. 170; Kummer, *Crelle*, Vol. XXVI, p. 268; Jacobi, *Crelle*, Vol. XXX, p. 46; Christoffel, *Crelle*, Vol. LXIII, p. 257; Bauer, *Crelle*, Vol. LXXI, p. 40; Borchardt, *Liouv. Journ.*, Vol. XII, p. 30; Sylvester, *Phil. Mag.*, Vol. II (1852), p. 138; Salmon, *Modern Higher Algebra*, Lesson VI; and see in particular Edward Smith, *Solution of the Equation of Secular Variation by a Method due to Hermite.* (Dissertation, University of Virginia. 1917.) Numerous other references are given in the paper last mentioned.

† Weierstrass, *Berlin. Monatsbericht*, May 18, 1868. Cf. also Kronecker, *Berlin. Monatsbericht* (1874), p. 1.

‡ Note the lemma of §§ 83, 84, and 85.

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81. If we form the system of equations (see [12] of preceding chapter)

$$[24] \qquad \phi_\lambda - e\psi_\lambda = 0 \quad (\lambda = 1, 2, \cdots, n),$$

then these equations may always be solved if their determinant vanishes.

This determinant is exactly the same as that in [23] if we write

$$\psi = \sum_{\lambda=1}^{\lambda=n} x_\lambda^2.$$

We assume that $e = k + li$, where $i = \sqrt{-1}$, and that we have found

$$x_\lambda = \xi_\lambda + \eta_\lambda i \quad (\lambda = 1, 2, \cdots, n)$$

as a system of values that satisfy the equations [24].

We must consequently have

$$\phi_\lambda(\xi_1 + \eta_1 i, \xi_2 + \eta_2 i, \cdots, \xi_n + \eta_n i) - (k + li)\psi_\lambda(\xi_1 + \eta_1 i, \xi_2 + \eta_2 i, \cdots, \xi_n + \eta_n i) = 0 \quad (\lambda = 1, 2, \cdots, n).$$

Since the real and the imaginary parts of these equations must of themselves be zero, it follows, when we observe that ϕ_λ and ψ_λ are linear functions of the variables, that

$$\phi_\lambda(\xi_1, \xi_2, \cdots, \xi_n) - k\psi_\lambda(\xi_1, \xi_2, \cdots, \xi_n) + l\psi_\lambda(\eta_1, \eta_2, \cdots, \eta_n) = 0,$$
$$\phi_\lambda(\eta_1, \eta_2, \cdots, \eta_n) - k\psi_\lambda(\eta_1, \eta_2, \cdots, \eta_n) - l\psi_\lambda(\xi_1, \xi_2, \cdots, \xi_n) = 0.$$

82. Next multiply these equations respectively by η_λ and ξ_λ, take the summation over them from 1 to n, and subtracting one of the resulting equations from the other, then, since (see [17] of the preceding chapter)

$$\sum_\lambda \eta_\lambda \phi_\lambda(\xi_1, \xi_2, \cdots, \xi_n) = \sum_\lambda \xi_\lambda \psi_\lambda(\eta_1, \eta_2, \cdots, \eta_n),$$
$$\sum_\lambda \eta_\lambda \psi_\lambda(\xi_1, \xi_2, \cdots, \xi_n) = \sum_\lambda \xi_\lambda \psi_\lambda(\eta_1, \eta_2, \cdots, \eta_n),$$

we have

$$l\left\{\sum_\lambda \eta_\lambda \psi_\lambda(\eta_1, \eta_2, \cdots, \eta_n) + \sum_\lambda \xi_\lambda \psi_\lambda(\xi_1, \xi_2, \cdots, \xi_n)\right\} = 0,$$

or,

$$[25] \qquad l\{\psi(\eta_1, \eta_2, \cdots, \eta_n) + \psi(\xi_1, \xi_2, \cdots, \xi_n)\} = 0.$$

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If it is possible to find systems of values of the quantities x_1, $x_2, \cdots, x_n$ which satisfy the equation [24] under the assumption that $e = k + li$, then these values must satisfy at the same time [25]; but since after our hypothesis the quantity within the brackets cannot vanish, it follows that l must be equal to zero; that is, every value of e for which the determinant vanishes, is real.

Hence we have the theorem:

In order that a quadratic form $\phi(x_1, x_2, \cdots, x_n)$ *be invariably positive, it is necessary and sufficient that the development of the determinant* [23] *which admits of only real roots, when expanded in powers of e, viz.*

$$[26] \qquad e^n - B_1 e^{n-1} + B_2 e^{n-2} - \cdots + (-1)^n B_n = 0,$$

consist of $n+1$ *terms and that these terms be alternately positive and negative.*

If the function is to be invariably negative, then the equation [26] *must be complete and have continuation of sign.*

Thus for the case where the variables are subjected to no conditions we have derived the criteria as to whether or not a homogeneous quadratic form is a definite one directly from the coefficients of the function and in a form that is perfectly symmetric.

83. Lemma. If a homogeneous function of the second degree $\psi(x_1, x_2, \cdots, x_n)$ can become zero for any system of real values of the variables which are not all zero, then ψ may be both positive and negative, it being presupposed that the determinant of ψ is different from zero.

Let the function ψ vanish for the system of values $(\xi_1, \xi_2, \cdots, \xi_n)$ and instead of $x_1, x_2, \cdots, x_n$ write in ψ the arguments $\xi_1 + c_1 k$, $\xi_2 + c_2 k, \cdots, \xi_n + c_n k$, where the c's are indeterminate constants.

Developing with respect to powers of k we have

$$\psi(\xi_1 + c_1 k, \xi_2 + c_2 k, \cdots, \xi_n + c_n k) = 2k \sum_{\alpha=1}^{\alpha=n} c_\alpha \psi_\alpha(\xi_1, \xi_2, \cdots, \xi_n) + k^2 \psi(c_1, c_2, \cdots, c_n). \qquad (i)$$

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By hypothesis the ξ's are not all zero, and the determinant of ψ being different from zero, it follows that $\psi_\alpha (\alpha = 1, 2, \cdots, n)$ cannot all be zero.

Since, furthermore, $c_\alpha (\alpha = 1, 2, \cdots, n)$ are arbitrary constants, we may so choose them that $\sum_{\alpha=1}^{\alpha=n} c_\alpha \psi_\alpha (\xi_1, \xi_2, \cdots, \xi_n)$ is *not* equal to zero.

Now by taking k sufficiently small we may cause the sign of the expression (i) to depend only upon the first term on the right-hand side of that expression.

Hence, if we choose k positive or negative, we have systems of values $(x_1, x_2, \cdots, x_n)$ which make ψ positive or negative.

84. The determinant of the system of equations [24] is formed from the partial derivatives of

$$\phi(x_1, x_2, \cdots, x_n) - e\psi(x_1, x_2, \cdots, x_n),$$

that is, from $\phi_\alpha(x_1, x_2, \cdots, x_n) - e\psi_\alpha(x_1, x_2, \cdots, x_n) = 0 \qquad (ii)$

$$(\alpha = 1, 2, \cdots, n),$$

where ϕ_α and ψ_α denote $\frac{1}{2}\frac{\partial \phi}{\partial x_\alpha}$ and $\frac{1}{2}\frac{\partial \psi}{\partial x_\alpha}$ respectively. If this determinant is equal to zero for a value of e, it follows that we can give to the variables $x_1, x_2, \cdots, x_n$ values that are not all zero and in such a way that the n equations (ii) exist. Let this value of e be $e = k + li$; then if $l \leqq 0$, it may be shown that the function ψ can have both positive and negative values.

Denote the system of values $(x_1, x_2, \cdots, x_n)$ which satisfy the equation (ii) by $x_\alpha = \xi_\alpha + i\eta_\alpha \quad (\alpha = 1, 2, \cdots, n)$;

then, as in § 82, it may be proved that

$$l[\psi(\xi_1, \xi_2, \cdots, \xi_n) + \psi(\eta_1, \eta_2, \cdots, \eta_n)] = 0. \qquad (iii)$$

Since by hypothesis l is not zero, the equation (iii) can only exist either when $\psi(\xi_1, \xi_2, \cdots, \xi_n)$ and $\psi(\eta_1, \eta_2, \cdots, \eta_n)$ have opposite values (and then it is proved, what we wish to show, that ψ can have both positive and negative values), or when the two values of the function are both zero (and then from what was seen in the preceding section ψ can take both positive and negative values).

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85. In this connection it is interesting to prove the following theorem: *If the determinant formed from the partial derivatives of the homogeneous quadratic form* $\psi(x_1, x_2, \cdots, x_n)$ *is different from zero, and if among the infinite number of quadratic forms*

$$\lambda\phi(x_1, x_2, \cdots, x_n) + \mu\psi(x_1, x_2, \cdots, x_n)$$

there is one definite quadratic form, the determinant formed from the partial derivatives of

$$\phi(x_1, x_2, \cdots, x_n) - e\psi(x_1, x_2, \cdots, x_n)$$

vanishes for only real values of e.

The theorem will also be true if the determinant of ϕ (and not as assumed of ψ) is different from zero.

Let $\lambda_1\phi + \mu_1\psi$ be a definite quadratic form, and write

$$\lambda_1\phi + \mu_1\psi = \overline{\psi}(x_1, x_2, \cdots, x_n).$$

We shall further choose two constants λ_0 and μ_0 in such a way that when we put

$$\lambda_0\phi + \mu_0\psi = \overline{\phi}(x_1, x_2, \cdots, x_n),$$

$\overline{\phi}$ is different from zero.

We know from the previous article that the determinant formed from the equations

$$\overline{\phi}_\alpha - k\overline{\psi}_\alpha = 0 \quad (\alpha = 1, 2, \cdots, n)$$

can only vanish for real values of k. The equations

$$\overline{\phi}_\alpha - k\overline{\psi}_\alpha = 0 \quad (\alpha = 1, 2, \cdots, n) \tag{iv}$$

may be written in the form

$$(\lambda_0 - k\lambda_1)\phi_\alpha + (\mu_0 - k\mu_1)\psi_\alpha = 0 \quad (\alpha = 1, 2, \cdots, n),$$

or

$$\phi_\alpha = \frac{k\mu_1 - \mu_0}{\lambda_0 - k\lambda_1}\psi_\alpha \quad (\alpha = 1, 2, \cdots, n). \tag{v}$$

If we eliminate $x_1, x_2, \cdots, x_n$ from these equations, we must have the same determinant for their solution as from the equations (*iv*).

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Hence every k which causes this last determinant to vanish must also cause the first determinant to vanish. But the k's are all real. It follows that if we form from them the n expressions

$$e = \frac{k\mu_1 - \mu_0}{\lambda_0 - k\lambda_1},$$

these quantities must also be real.

Hence the determinant of the n equations

$$\phi_\alpha - e\psi_\alpha = 0 \quad (\alpha = 1, 2, \cdots, n)$$

has always n real roots e.

We may therefore say: *If among all the quadratic forms which are contained in the form*

$$\lambda\phi(x_1, x_2, \cdots, x_n) + \mu\psi(x_1, x_2, \cdots, x_n),$$

there is one which can have only positive or only negative values, then the determinant of $\phi - e\psi$ will have only real roots, it being assumed that the determinant of ϕ or of ψ is not zero.

The theorem in § 80 is accordingly proved in its greatest generality.

86. The case where equations of condition are present may be easily reduced to the case already considered. The determinant [20] was the result of eliminating the quantities $x_1, x_2, \cdots, x_n$, $e_1, e_2, \cdots, e_m$ from the $n + m$ equations

$$[18] \quad \sum_{\mu=1}^{\mu=n} A_{\lambda\mu}x_\mu - ex_\lambda + \sum_{\rho=1}^{\rho=m} e_\rho a_{\rho\lambda} = 0 \quad (\lambda = 1, 2, \cdots, n),$$

$$[15] \quad \theta_\rho = \sum_{\mu=1}^{\mu=n} a_{\rho\mu}x_\mu = 0 \quad (\rho = 1, 2, \cdots, m).$$

Since the result of the elimination is independent of the way in which it has been effected, we may first consider m of the quantities x, say: $x_1, x_2, \cdots x_m$, expressed by means of the equations [15] in terms of the remaining $n - m$ of the x's, which may be denoted by $\xi_1, \xi_2, \cdots, \xi_{n-m}$. We thus have

$$[27] \quad x_\mu = \sum_{\nu=1}^{\nu=n-m} C_{\mu\nu}\xi_\nu \quad (\mu = 1, 2, \cdots, m).$$

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Through the substitution of these values, let $\phi(x_1, x_2, \cdots, x_n)$ be transformed into $\bar{\phi}(\xi_1, \xi_2, \cdots, \xi_{n-m})$ and the equation

$$\sum_{\lambda=1}^{\lambda=n} x_\lambda^2 = 1 \text{ into } \psi(\xi_1, \xi_2, \cdots, \xi_{n-m}) = 1.$$

The function ψ is invariably positive and is only equal to zero when the variables themselves all vanish.

The equations [18] may be written in the form:

$$\frac{1}{2}\frac{\partial \phi}{\partial x_\lambda} - e x_\lambda + \sum_{\rho=1}^{\rho=m} e_\rho \frac{\partial \theta_\rho}{\partial x_\lambda} = 0 \quad (\lambda = 1, 2, \cdots, n).$$

Multiplying these equations respectively by $\frac{\partial x_\lambda}{\partial \xi_\nu}$ $(\lambda = 1, 2, \cdots, n)$, and adding the results, then, since

$$\sum_{\lambda=1}^{\lambda=n} \frac{\partial \theta_\rho}{\partial x_\lambda}\frac{\partial x_\lambda}{\partial \xi_\nu} = \frac{\partial \theta_\rho}{\partial \xi_\nu},$$

$$\sum_{\lambda=1}^{\lambda=n} \frac{\partial \phi}{\partial x_\lambda}\frac{\partial x_\lambda}{\partial \xi_\nu} = \frac{\partial \phi}{\partial \xi_\nu},$$

$$\sum_{\lambda=1}^{\lambda=n} x_\lambda \frac{\partial x_\lambda}{\partial \xi_\nu} = \frac{1}{2}\frac{\partial}{\partial \xi_\nu}\sum_{\lambda=1}^{\lambda=n} x_\lambda^2,$$

we have the following equations:

$$\frac{1}{2}\frac{\partial \phi}{\partial \xi_\nu} - \frac{1}{2} e \frac{\partial}{\partial \xi_\nu}\sum_{\lambda=1}^{\lambda=n} x_\lambda^2 + \sum_{\rho=1}^{\rho=m} e_\rho \frac{\partial \theta_\rho}{\partial \xi_\nu} = 0 \quad (\nu = 1, 2, \cdots, n).$$

The last term of this equation drops out if we substitute in it the expressions [27], since the θ_ρ expressed in the ξ's vanish identically, and we have the equations

$$[28] \qquad \frac{\partial \bar{\phi}}{\partial \xi_\nu} - e\frac{\partial \psi}{\partial \xi_\nu} = 0 \quad (\nu = 1, 2, \cdots, n - m).$$

Now give ν all values from 1 to $n - m$, and we have a system of $n - m$ linear homogeneous equations, from which we may eliminate the yet remaining $\xi_1, \xi_2, \cdots, \xi_{n-m}$. The result of this elimination is an equation in e and must give the same roots in e as [20]. The

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equations [28] are, however, created in exactly the same manner as the equations [24]. If, then, Δe is the determinant of these equations, it follows that the roots of the equation $\Delta e = 0$ are all real.

87. As the solution of the theorem proposed in § 77 the final result is:

In order that the homogeneous function of the second degree

$$\phi(x_1, x_2, \cdots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\lambda x_\mu,$$

$$A_{\lambda\mu} = A_{\mu\lambda},$$

be invariably positive for all systems of values of the quantities $x_1, x_2, \cdots, x_n$, *which satisfy the m linear homogeneous equations of condition*

$$\theta_\rho = \sum_{\mu=1}^{\mu=n} a_{\rho\mu} x_\mu = 0 \quad (\rho = 1, 2, \cdots, m),$$

it is necessary and sufficient that the form of the equation [20], *developed with respect to powers of e and which has only real roots, consist of* $n - m + 1$ *terms and that the signs associated with these terms be alternately positive and negative. There must, however, be only a continuation of sign if* ϕ *is to be invariably negative.*

The above method was first discovered by Lagrange, who did not, however, sufficiently emphasize the reality of the roots of equation [20].

II. APPLICATION OF THE CRITERIA JUST FOUND TO THE PROBLEM OF THIS CHAPTER

88. We have determined the exact conditions necessary for a homogeneous quadratic form to be definite for the case where the variables are to satisfy equations of condition and in a manner entirely symmetric in the coefficients of the given function together with those of the given equations of condition.

At the same time with the solution of this problem, the problem of maxima and minima which we have proposed in this chapter is solved.

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89. Having regard to the remarks made in § 71 and § 74 we have as a final result of our investigations the following theorem:

THEOREM. *If those positions are to be found on which a given regular function* $F(x_1, x_2, \cdots, x_n)$ *has a maximum or minimum value under the condition that the* n *variables* $x_1, x_2, \cdots, x_n$ *satisfy the* m *equations*

$$[a] \qquad f_\lambda(x_1, x_2, \cdots, x_n) = 0 \quad (\lambda = 1, 2, \cdots, m),$$

where f_λ *are likewise regular functions, we write*

$$[b] \qquad F + \sum_{\rho=1}^{\rho=m} e_\rho f_\rho = G(x_1, x_2, \cdots, x_n),$$

and seek the system of real values

$$x_1, x_2, \cdots, x_n, \quad e_1, e_2, \cdots, e_m$$

which satisfy the $n + m$ *equations*

$$[c] \qquad \begin{cases} \dfrac{\partial G}{\partial x_\lambda} = 0 & (\lambda = 1, 2, \cdots, n), \\ f_\lambda = 0. & (\lambda = 1, 2, \cdots, m). \end{cases}$$

If $(a_1, a_2, \cdots, a_n)$ *is such a system of values of* $x_1, x_2, \cdots, x_n$, *then we develop the difference*

$$G(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) - G(a_1, a_2, \cdots, a_n)$$

with respect to powers of h, *and have (since no terms of the first dimension can appear, owing to equations* [c]) *the following development:*

$$[d] \quad G(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) - G(a_1, a_2, \cdots, a_n) = \frac{1}{2} \sum_{\mu, \nu} G_{\mu\nu}(a_1, a_2, \cdots, a_n) h_\mu h_\nu + \cdots.$$

We must next see whether the function

$$[e] \qquad \phi(h_1, h_2, \cdots, h_n) = \sum_{\mu, \nu} G_{\mu\nu} h_\mu h_\nu$$

is invariably positive or invariably negative for all systems of values of the h*'s which satisfy the* m *equations*

$$[f] \qquad \sum_{\mu=1}^{\mu=n} f_{\rho\mu}(a_1, a_2, \cdots, a_n) h_\mu = 0 \quad (\rho = 1, 2, \cdots, m).$$

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To do this we form the determinant

$$[g] \quad \begin{vmatrix} G_{11}-e, & G_{12}, & \cdots, & G_{1n}, & f_{11}, & f_{21}, & \cdots, & f_{m1} \\ G_{21}, & G_{22}-e, & \cdots, & G_{2n}, & f_{12}, & f_{22}, & \cdots, & f_{m2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{n1}, & G_{n2}, & \cdots, & G_{nn}-e, & f_{1n}, & f_{2n}, & \cdots, & f_{mn} \\ f_{11}, & f_{12}, & \cdots, & f_{1n}, & 0, & 0, & \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{m1}, & f_{m2}, & \cdots, & f_{mn}, & 0, & 0, & \cdots, & 0 \end{vmatrix},$$

and this determinant put equal to 0 *is an equation of the* $m-n$ *degree in* e, *which has only real roots. Developing the determinant with respect to powers of* e, *we have to see whether the development consists of* $n-m+1$ *terms with alternately positive and negative sign or with only continuation of sign.*

If the first is the case, the function ϕ *is invariably positive, and the function* F *has on the position* $(a_1, a_2, \cdots, a_n)$ *a* **minimum** *value; if, on the contrary, the latter is true, then* ϕ *is invariably negative, and* F *has on the position* $(a_1, a_2, \cdots, a_n)$ *a* **maximum** *value.*

This criterion fails, however, when ϕ vanishes identically, because the quantities $G_{\mu\nu}$ vanish for the position $(a_1, a_2, \cdots, a_n)$; and it also fails when the smallest or greatest root of $\Delta e = 0$ is zero, since in this case we may always so choose the h's that ϕ vanishes without the h's being all identically zero (see § 83). In the latter case the function $\phi(x)$ is an indefinite or a semi-definite form (§ 78).

In both of these cases the development $[d]$ begins with terms of the third or higher dimensions, and for the same reason as that stated at the end of § 63 we cannot assert that in general a maximum or minimum will enter on the position $(a_1, a_2, \cdots, a_n)$.

90. We give next two geometrical examples illustrating the above principles.

PROBLEM I. *Determine the greatest and the smallest curvature at a regular point of a surface* $F(x, y, z) = 0$.

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If at a regular point P of a plane curve we draw a tangent and from a neighboring point P' on the curve we drop a perpendicular $P'Q$ upon this tangent, then the value that $2\frac{P'Q}{\overline{PP'}^2} = \frac{\Delta\phi}{\Delta s}$ approaches, if we let P' come indefinitely near P, is called the *curvature* of the curve at the point P. If the curve is a circle with radius r, the above ratio approaches $\frac{1}{r}$ as a limiting value and is, therefore, the same for all points of the circle. Now construct the *osculating* circle which passes through the two neighboring points P and P' of the given curve. The arc of the circle PP' may be put equal to the arc PP' of the curve, when P and P' are taken very near each other, and consequently, if r is the radius of this circle, the curvature of the curve is determined through the formula

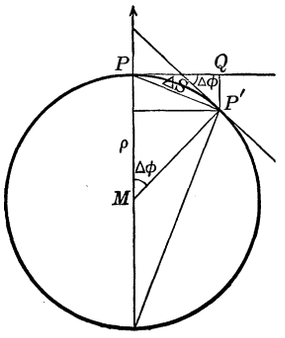

FIG. 11

$$[1] \qquad \frac{2\,P'Q}{\overline{PP'}^2} = \frac{1}{r}.$$

The quantity r is called the *radius of curvature*, and the center M of the circle which lies on the normal drawn to the curve at the point P is known as the *center of curvature* at the point P. The curvature is counted positive or negative according as the line $P'Q$, or, what amounts to the same thing, MP has the same or opposite direction as that direction of the normal which has been chosen positive.

If we have a given surface and if the normal at any regular point of this surface is drawn, then every plane drawn through this normal will cut the surface in a curve which has at the point P a definite tangent and a definite curvature in the sense given above.

The *curvature* of this curve at the point P is called the *curvature* of the surface at the point $P=(x, y, z)$ in the direction of the tangent which is determined through the normal section in question.

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Following the definitions given above it is easy to fix the analytic conception of the curvature of a surface and then to formulate the problem in an analytic manner.

If $P'=(x',\ y',\ z')$ is a neighboring point of P on the surface, the equation of the surface may be written in the form:

$$[2]\quad \begin{aligned} 0 = {} & F_1(x'-x)+F_2(y'-y)+F_3(z'-z) \\ & +\tfrac{1}{2}\{F_{11}(x'-x)^2+F_{22}(y'-y)^2+F_{33}(z'-z)^2 \\ & +2F_{12}(x'-x)(y'-y)+2F_{23}(y'-y)(z'-z) \\ & +2F_{31}(z'-z)(x'-x)\}+\cdots, \end{aligned}$$

where
$$F_1=\frac{\partial F}{\partial x},\quad F_2=\frac{\partial F}{\partial y},\quad F_3=\frac{\partial F}{\partial z},$$
$$F_{11}=\frac{\partial^2 F}{\partial x^2},\quad F_{22}=\frac{\partial^2 F}{\partial y^2},\quad F_{33}=\frac{\partial^2 F}{\partial z^2},$$
$$F_{12}=\frac{\partial^2 F}{\partial x\partial y},\quad F_{23}=\frac{\partial^2 F}{\partial y\partial z},\quad F_{31}=\frac{\partial^2 F}{\partial z\partial x}.$$

The equation of the tangential plane at the point P is

$$[3]\qquad F_1(\xi-x)+F_2(\eta-y)+F_3(\zeta-z)=0,$$

where $\xi,\ \eta,\ \zeta$ are the running coördinates.

Therefore, if we write for brevity

$$[4]\qquad \sqrt{F_1^2+F_2^2+F_3^2}=H,$$

and take as the positive direction of the normal of the surface at the point P that direction for which H is positive, then the direction-cosines of this normal are

$$\frac{F_1}{H},\quad \frac{F_2}{H},\quad \text{and}\quad \frac{F_3}{H}.$$

Consequently the distance from P' to the tangential plane is

$$[5]\qquad P'Q=\frac{F_1}{H}(x'-x)+\frac{F_2}{H}(y'-y)+\frac{F_3}{H}(z'-z).$$

The negative or positive sign is to be given to the expression on the right-hand side according as the length $P'Q$ has the same

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or opposite direction as that direction of the normal which has been chosen positive.

In the first case, paying attention to [2], which has to be satisfied, since P' lies upon the surface, we have

$$[6] \qquad \frac{2P'Q}{\overline{PP'}^2}$$

$$= \frac{F_{11}(x'-x)^2+F_{22}(y'-y)^2+F_{33}(z'-z)^2+2F_{12}(x'-x)(y'-y)+\cdots}{HS^2},$$

where $$S^2=(x'-x)^2+(y'-y)^2+(z'-z)^2.$$

In the case where the direction $P'Q$ is contrary to the positive direction of the normal, we must give the negative sign to the right-hand side of [6].

Now let P' approach nearer and nearer P; then the quantities

$$\frac{x'-x}{S}, \quad \frac{y'-y}{S}, \quad \frac{z'-z}{S},$$

which represent the direction-cosines of the line PP', become the direction-cosines of the tangent at the point P of the normal section that is determined through P'. Representing these by α, β, γ and the limiting value of $2\frac{P'Q}{\overline{PP'}^2}$ by κ, then

$$[7] \quad \kappa=\frac{1}{H}\{F_{11}\alpha^2+F_{22}\beta^2+F_{33}\gamma^2+2F_{12}\alpha\beta+2F_{23}\beta\gamma+2F_{31}\gamma\alpha\},$$

where the terms of higher degree in $x'-x$, etc. are neglected. In this formula κ represents the *curvature of the surface in the direction determined by* α, β, γ. This is to be taken positive or negative according as the direction of the length MP, where M is the center of curvature, corresponds to the positive direction or not.

If the coördinates of the center of curvature are represented by x_0, y_0, z_0 and the radius of curvature by ρ, then

$$x-x_0=\rho\frac{F_1}{H};$$

or, since $$\kappa=\frac{1}{\rho},$$

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$$[8] \qquad \begin{cases} x - x_0 = \dfrac{F_1}{F_{11}\alpha^2 + F_{22}\beta^2 + \cdots + 2F_{31}\gamma\alpha}, \\ y - y_0 = \dfrac{F_2}{F_{11}\alpha^2 + F_{22}\beta^2 + \cdots + 2F_{31}\gamma\alpha}, \\ z - z_0 = \dfrac{F_3}{F_{11}\alpha^2 + F_{22}\beta^2 + \cdots + 2F_{31}\gamma\alpha}. \end{cases}$$

Since H does not appear in these expressions, we see that the position of the center of curvature is independent of the choice of the direction of the normal.

Suppose that the normal plane which is determined through the direction α, β, γ is turned about the normal until it returns to its original position. Then, while α, β, γ vary in a definite manner, the function κ of α, β, γ assumes different values at every instance, and since it is a regular function, it must have a maximum value for a definite system of values (α, β, γ) and likewise also a minimum value for another definite system of values (α, β, γ).

The quantity $\dfrac{1}{H}$ has the same value for all normal sections that are laid through the same normal.* We have, therefore, to seek the systems of values (α, β, γ) for which the expression

$$F_{11}\alpha^2 + F_{22}\beta^2 + F_{33}\gamma^2 + 2F_{12}\alpha\beta + 2F_{23}\beta\gamma + 2F_{31}\gamma\alpha$$

assumes its greatest and its smallest value.

We have also to observe that the variables α, β, γ must satisfy the equations of condition

$$[9] \qquad \begin{cases} F_1\alpha + F_2\beta + F_3\gamma = 0, \\ \alpha^2 + \beta^2 + \gamma^2 = 1, \end{cases}$$

the first of which says that the direction which is determined through α, β, γ is to lie in the tangential plane of the surface at the point P, while the second equation is the well-known relation among the direction-cosines of a straight line in space.

*See Salmon, *A Treatise on the Analytic Geometry of Three Dimensions* (Fourth Edition), p. 259.

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Following the methods indicated in § 89, we write

$$[10] \quad G = F_{11}\alpha^2 + F_{22}\beta^2 + \cdots + 2F_{31}\gamma\alpha - e(\alpha^2+\beta^2+\gamma^2-1) + 2e'(F_1\alpha + F_2\beta + F_3\gamma),$$

and we then have (§ 89, [c]) to form the equations

$$\frac{\partial G}{\partial \alpha} = 0, \quad \frac{\partial G}{\partial \beta} = 0, \quad \frac{\partial G}{\partial \gamma} = 0,$$
$$F_1\alpha + F_2\beta + F_3\gamma = 0,$$

from which we must eliminate α, β, γ, and e'.

These equations are

$$[11] \quad \begin{cases} (F_{11}-e)\alpha + F_{12}\beta + F_{13}\gamma + F_1 e' = 0, \\ F_{21}\alpha + (F_{22}-e)\beta + F_{23}\gamma + F_2 e' = 0, \\ F_{31}\alpha + F_{32}\beta + (F_{33}-e)\gamma + F_3 e' = 0, \\ F_1\alpha + F_2\beta + F_3\gamma = 0, \end{cases}$$

where $F_{\lambda\mu} = F_{\mu\lambda} \quad (\lambda, \mu = 1, 2, 3)$.

Through elimination we have

$$[12] \quad \begin{vmatrix} F_{11}-e, & F_{12}, & F_{13}, & F_1 \\ F_{21}, & F_{22}-e, & F_{23}, & F_2 \\ F_{31}, & F_{32}, & F_{33}-e, & F_3 \\ F_1, & F_2, & F_3, & 0 \end{vmatrix} = 0.$$

This is an equation of the second degree in e, and consequently gives us two values e_1 and e_2, which are maximum and minimum values, since both maximum and minimum values enter, as shown above. Multiplying the first three equations [11] by α, β, γ respectively and adding the results, we have

$$[13] \quad F_{11}\alpha^2 + F_{22}\beta^2 + F_{33}\gamma^2 + 2F_{12}\alpha\beta + 2F_{23}\beta\gamma + 2F_{31}\gamma\alpha = e.$$

Hence, from [7] we have

$$[14] \quad \kappa = \frac{1}{\rho} = \frac{e}{H}.$$

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Consequently the two principal curvatures at the point P have the values

$$[15] \qquad \begin{cases} \dfrac{1}{\rho_1} = \dfrac{e_1}{H}, \\ \dfrac{1}{\rho_2} = \dfrac{e_2}{H}, \end{cases}$$

and the coördinates of the corresponding centers of curvature are found from the formulæ

$$[16] \qquad \begin{cases} x - x_{01} = \dfrac{F_1}{e_1}, \quad y - y_{01} = \dfrac{F_2}{e_1}, \quad z - z_{01} = \dfrac{F_3}{e_1}, \\ x - x_{02} = \dfrac{F_1}{e_2}, \quad y - y_{02} = \dfrac{F_2}{e_2}, \quad z - z_{02} = \dfrac{F_3}{e_2}. \end{cases}$$

In order to determine e, let us write

$$D_{11} = (F_{22} - e)(F_{33} - e) - F_{23}^2,$$
$$D_{12} = F_{23}F_{13} - F_{12}(F_{33} - e),$$

and form from these the corresponding quantities through the cyclic interchange of the indices. Equation [12] may be written in the form*

$$D_{11}F_1^2 + D_{22}F_2^2 + D_{33}F_3^2 + 2D_{12}F_1F_2 + 2D_{23}F_2F_3 + 2D_{31}F_3F_1 = 0.$$

Developing this expression with respect to powers of e, we have

$$[17] \qquad H^2e^2 - Le + M = 0,$$

where $$L = H^2(F_{11} + F_{22} + F_{33}) - (F_{11}F_1^2 + F_{22}F_2^2 + F_{33}F_3^2) + 2F_{12}F_1F_2 + 2F_{23}F_2F_3 + 2F_{31}F_3F_1$$

and $$M = (F_{22}F_{33} - F_{23}^2)F_1^2 + (F_{33}F_{11} - F_{31}^2)F_2^2 + (F_{11}F_{22} - F_{12}^2)F_3^2 + (F_{12}F_{13} - F_{23}F_{11})F_2F_3 + (F_{23}F_{21} - F_{31}F_{22})F_3F_1 + (F_{31}F_{32} - F_{12}F_{33})F_1F_2.$$

From [17] we have at once the values of the sum and the product of the two principal curvatures, viz. (see equation [15]):

$$[18] \qquad \begin{cases} \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} = \dfrac{L}{H^3}, \\ \dfrac{1}{\rho_1\rho_2} = \dfrac{M}{H^4}. \end{cases}$$

* See Salmon, loc. cit., p. 257.

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We have thus expressed the sum of the reciprocal radii of curvature and also the measure of curvature of the surface at the point P directly through the coördinates of this point.

Although the formulæ are somewhat complicated, they are used extensively and with great advantage.

In the case of *minimal surfaces*,* which are characterized through the equation

$$\rho_1 + \rho_2 = 0,$$

we have

$$L = 0.$$

This is therefore the general differential equation for minimal surfaces.

91. PROBLEM II. *From a given point* (a, b, c) *to a given surface* $F(x, y, z) = 0$ *draw a straight line whose length is a maximum or a minimum.*

Write $$G = (x-a)^2 + (y-b)^2 + (z-c)^2 + 2\lambda F(x, y, z). \qquad (i)$$

Then the quantities x, y, z, λ are to be determined (see § 89, [c]) from the following equations:

$$\left.\begin{aligned} x - a + \lambda F_1 &= 0, \\ y - b + \lambda F_2 &= 0, \\ z - c + \lambda F_3 &= 0, \\ F(x, y, z) &= 0. \end{aligned}\right\} \qquad (ii)$$

It follows, since F_1, F_2, F_3 are proportional to the direction-cosines of the normal to the surface at the point (x, y, z), that the points determined through these equations are such that lines joining them to the point (a, b, c) stand normal to the surface.

If $P = (x, y, z)$ is such a point, then to determine whether for this point the quantity

$$(x-a)^2 + (y-b)^2 + (z-c)^2$$

is in reality a maximum or a minimum, we substitute $x+u$, $y+v$, $z+w$ instead of x, y, z in the function G. The quantities u, v, w are, of course, taken very small.

*See papers by the author on this subject in the first numbers of the *Mathematical Review*.

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We must develop the difference

$$G(x+u,\ y+v,\ z+w)-G(x,\ y,\ z) \qquad (iii)$$

in powers of u, v, and w.

The terms of the first dimension drop out, and the aggregate of the terms of the second dimension is

$$\psi = u^2+v^2+w^2+\lambda(F_{11}u^2+F_{22}v^2+F_{33}w^2 + 2F_{12}uv+2F_{23}vw+2F_{31}wu). \qquad (iv)$$

Since the point $(x+u,\ y+v,\ z+w)$ must also lie upon the surface, the quantities u, v, w must satisfy the condition

$$F_1u+F_2v+F_3w=0, \qquad (v)$$

where the terms of the higher dimensions are omitted (see [8] of the present chapter).

If we wish to determine whether the function ψ is invariably positive or invariably negative for all systems of values $(u,\ v,\ w)$ which satisfy equation (v), we may seek the minimum or maximum of this function ψ under the condition that the variables are limited, besides the equation (v), to the further restriction (cf. [16] of § 77) that

$$u^2+v^2+w^2-1=0. \qquad (vi)$$

For this purpose we form the function

$$\psi - e(u^2+v^2+w^2-1)+2e'(F_1u+F_2v+F_3w), \qquad (vii)$$

and writing its partial derivatives with respect to u, v, and w equal to zero, we derive the equations

$$\left.\begin{aligned} \left(F_{11}-\frac{e-1}{\lambda}\right)u+F_{12}v+F_{13}w+\frac{e'}{\lambda}F_1&=0,\\ F_{21}u+\left(F_{22}-\frac{e-1}{\lambda}\right)v+F_{23}w+\frac{e'}{\lambda}F_2&=0,\\ F_{31}u+F_{32}v+\left(F_{33}-\frac{e-1}{\lambda}\right)w+\frac{e'}{\lambda}F_3&=0. \end{aligned}\right\} \qquad (viii)$$

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Eliminating $u, v, w, \frac{e'}{\lambda}$ from equations (v) and $(viii)$, we have here exactly the same system of equations as in [12] of the preceding problem, except that here $\frac{e-1}{\lambda}$ and e' stand in the place of e and e'.

Denote the two roots of the quadratic equation in e, which is the result of the above elimination, by e_1 and e_2, and the corresponding radii of curvature of the normal sections by ρ_1 and ρ_2; then, since $\frac{e-1}{\lambda}$ has the same meaning as e in the previous problem,

$$\frac{1}{\rho_1} = \frac{e_1-1}{\lambda} \cdot \frac{1}{H},$$
$$\frac{1}{\rho_2} = \frac{e_2-1}{\lambda} \cdot \frac{1}{H},$$

where the positive direction of the normal to the surface is so chosen that $H>0$.

If for the position (x, y, z) a minimum of the distance is to enter, then both values of the e must be positive; if a maximum, then e_1 and e_2 must be negative.

It is easy to give a geometric interpretation of this result: Let PN be the positive direction of the normal and $A=(a, b, c)$. Then from (ii) it follows that the length from A to P has the same or opposite direction as PN, according as λ is negative or positive.

Hence, from (ii),

$$AP = -\lambda H.$$

If the centers of curvature corresponding to ρ_1 and ρ_2 be denoted by M_1 and M_2, then

$$M_1P = \frac{\lambda H}{e_1-1} = -\frac{AP}{e_1-1},$$
$$M_2P = \frac{\lambda H}{e_2-1} = -\frac{AP}{e_2-1}.$$

Hence $$e_1 = \frac{M_1A}{M_1P} \quad \text{and} \quad e_2 = \frac{M_2A}{M_2P}.$$

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If, then, M_1 and M_2 lie on the same side of P and if A lies between M_1 and M_2, as in Figs. 12 and 13, then the e's have different signs and there is neither a maximum nor a minimum.

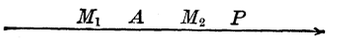

FIG. 12

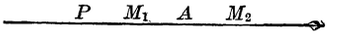

FIG. 13

If M_1 and M_2 lie on the same side of P while A is without the interval $M_1 \cdots M_2$, then a minimum or maximum will enter according as A starting from one of the centers of curvature lies upon the same side as P or not (see Figs. 14 and 15).

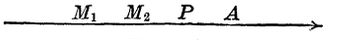

FIG. 14

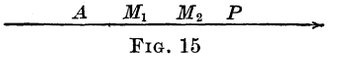

FIG. 15

If the points M_1 and M_2 lie on different sides of P and if A is situated within the interval $M_1 \cdots M_2$, as in Fig. 16, then there is always a minimum. If, however, A lies without the interval $M_1 \cdots M_2$, then there is neither a maximum nor a minimum.

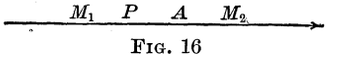

FIG. 16

In whatever manner M_1 and M_2 may lie, if A coincides with one of these points, then one of the two values of e is equal to zero, and the general remark stated at the end of § 89 is applicable.

The above results are derived in a different manner by Goursat, *Cours D'Analyse*, Vol. I, p. 118.

The case may also happen here (see § 72) that in the solution of the equations (*ii*) and (*iii*) a singular point of the surface is found at the point P, at which $F_1 = 0 = F_2 = F_3$. We cannot proceed as above, since, there being no definite normal of the surface at such a point, the determination whether for this point a maximum or minimum really exist cannot be decided in the manner we have just given.

The general remark of § 73 indicates how we are to proceed.

92. Brand's problems. The two following problems taken from the theory of light were prepared by my colleague, Professor Louis Brand.

PROBLEM I. *Reflection at the surface* $F(x, y, z) = 0$. *A ray passes from a point* P_1 *to a point* P *on a given surface and is reflected to a point* P_2. *When is* $P_1P + PP_2$ *a minimum?*

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Write $PP_1 = d_1$ and $PP_2 = d_2$ so that

$$d_i = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2} \quad (i = 1, 2).$$

We seek to find the condition that makes $d_1 + d_2$ an extreme when P is subjected to the condition of lying on the surface

[1] $$F(x, y, z) = 0.$$

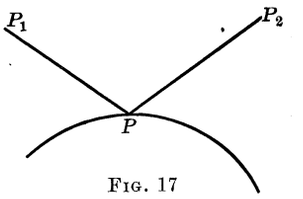

FIG. 17

Using the Lagrangian method (§ 89) we must find the extremes of the function

$$\phi(x, y, z) = d_1 + d_2 + \lambda F(x, y, z).$$

Writing ϕ_x for $\dfrac{\partial \phi(x, y, z)}{\partial x}$ etc., the necessary conditions for an extreme, viz., $\phi_x = \phi_y = \phi_z = 0$, give

[2] $$\begin{cases} \dfrac{x_1 - x}{d_1} + \dfrac{x_2 - x}{d_2} = \lambda F_x, \\ \dfrac{y_1 - y}{d_1} + \dfrac{y_2 - y}{d_2} = \lambda F_y, \\ \dfrac{z_1 - z}{d_1} + \dfrac{z_2 - z}{d_2} = \lambda F_z. \end{cases}$$

Let the direction-cosines of the lines PP_1 and PP_2 be l_1, m_1, n_1 and l_2, m_2, n_2 respectively; and let l, m, n denote the direction-cosines of the normal to the surface [1] at the point P. Furthermore, since F_x, F_y, F_z are proportional to l, m, n, write

$$\lambda F_x = kl, \quad \lambda F_y = km, \quad \lambda F_z = kn.$$

Equations [2] then become

[3] $$\begin{cases} l_1 + l_2 = kl, \\ m_1 + m_2 = km, \\ n_1 + n_2 = kn. \end{cases}$$

Designate the angle between PP_1 and PP_2 by $(1, 2)$; between PP_1 and the normal by $(1, n)$; between PP_2 and the normal by $(2, n)$. It is seen then that

$$\cos(1, 2) = l_1 l_2 + m_1 m_2 + n_1 n_2,$$
$$\cos(1, n) = l_1 l + m_1 m + n_1 n,$$
$$\cos(2, n) = l_2 l + m_2 m + n_2 n.$$

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Multiplying equations [3] by l_1, m_1, n_1, respectively, and adding, it follows, since $l_1^2 + m_1^2 + n_1^2 = 1$, that

[4] $$1 + \cos(1, 2) = k \cos(1, n).$$

Similarly, by multiplying equations [3] by l_2, m_2, n_2, respectively, and adding, we get

[5] $$\cos(1, 2) + 1 = k \cos(2, n).$$

From [4] and [5] we have

$$\cos(1, n) = \cos(2, n), \text{ or}$$

[6] $$(1, n) = (2, n).$$

Moreover, upon multiplying equations [3] by l, m, n, respectively, and adding, we get

$$\cos(1, n) + \cos(2, n) = k, \text{ or, from [6]},$$
$$k = 2 \cos(1, n).$$

Substituting this value of k in [4], we have

$$1 + \cos(1, 2) = 2 \cos^2(1, n),$$

so that $\cos(1, 2) = 2 \cos^2(1, n) - 1 = \cos 2(1, n) = \cos 2(2, n)$.

It follows that $(1, 2) = 2(1, n) = 2(2, n)$,

and that the lines PP_1, PP_2, and the normal must lie in the same plane, and it is further seen that the normal bisects the angle between PP_1 and PP_2.

We have thus arrived at the condition which is an optical law: *The incident and reflected rays must lie in a normal plane, and the angle of incidence must be equal to the angle of reflection.*

The above result is merely a *necessary* condition for an extreme; to find whether an extreme really exists, and if it does, whether it is a maximum or a minimum, let us choose the plane P_1PP_2 as the xy-plane.

If the curve cut from the surface by the plane P_1PP_2 has the equation

[7] $$y = f(x),$$

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the problem now becomes to determine the nature of the point P which makes $\frac{d\psi(x)}{dx} = 0$, where

$$\psi(x) = d_1 + d_2 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2} + \sqrt{(x_2 - x)^2 + (y_2 - y)^2},$$

the y being replaced by $f(x)$.

Now $$\frac{d\psi}{dx} = \frac{(x - x_1) + (y - y_1)y'}{d_1} + \frac{(x - x_2) + (y - y_2)y'}{d_2},$$

while the equation of the normal to the curve [7] at $P(x, y)$ is

$$(x - \xi) + (y - \eta)\, y'_P = 0,$$

and the distance of the point (x_i, y_i) from this normal is

$$h_i = \frac{(x - x_i) + (y - y_i)y'_P}{\sqrt{1 + y'^2_P}}.$$

Further, take the origin at the point P and the tangent to the surface at P lying in the plane P_1PP_2 as the x-axis. Then $\frac{d\psi}{dx} = 0$ shows that

$$\frac{h_1}{d_1} + \frac{h_2}{d_2} = 0,$$

and as h_1 and h_2 have opposite signs, since P_1 and P_2 lie upon opposite sides of the normal,

$$\sin(1, n) = \sin(2, n),$$

or
$$(1, n) = (2, n),$$

as stated before in [6].

Note that

$$\frac{d^2\psi}{dx^2} = \frac{d_1[1 + y'^2 + (y - y_1)y''] - \frac{1}{d_1}[(x - x_1) + (y - y_1)y']^2}{d_1^2} + \frac{d_2[1 + y'^2 + (y - y_2)y''] - \frac{1}{d_2}[(x - x_2) + (y - y_2)y']^2}{d_2^2}.$$

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It follows that for the origin and the direction $y' = 0$,

$$\frac{d^2\psi}{dx^2} = \frac{d_1(1 - y_1 y'') - \frac{x_1^2}{d_1}}{d_1^2} + \frac{d_2(1 - y_2 y'') - \frac{x_2^2}{d_2}}{d_2^2}.$$

Writing $\theta = (1, n) = (2, n)$,

we note that $\frac{y_1}{d_1} = \frac{y_2}{d_2} = \cos\theta$,

$$-\frac{x_1}{d_1} = \frac{x_2}{d_2} = \sin\theta,$$

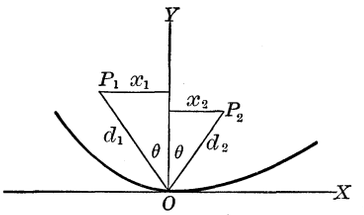

FIG. 18

so that $$\frac{d^2\psi}{dx^2} = \left(\frac{1}{d_1} + \frac{1}{d_2}\right)\cos^2\theta - 2y''\cos\theta.$$

From this it is seen that

$$\frac{d^2\psi}{dx^2} \gtreqless 0 \text{ according as } y'' \gtreqless \frac{1}{2}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\cos\theta.$$

Since $y' = 0$, we note that y'' is the curvature of curve $y = f(x)$ at the origin, that is, $y'' = \frac{1}{\rho}$, where ρ is the radius of curvature. Hence, when

[8] $$\frac{1}{\rho} < \frac{1}{2}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\cos\theta,$$

$\frac{d^2\psi}{dx^2} > 0$, and the path is a minimum; when

[9] $$\frac{1}{\rho} > \frac{1}{2}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\cos\theta,$$

$\frac{d^2\psi}{dx^2} < 0$, and the path is a maximum.

To interpret this result geometrically it is seen that

$$\frac{1}{2}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\cos\theta$$

is the curvature of the ellipse whose foci are at P_1 and P_2 and which passes through P (see Pascal, *Repertorium der Höheren Mathematik*, Vol. II, 1, p. 245). The quantities d_1 and d_2 are its focal radii at P, and θ is the angle between either focal radius and the normal to the ellipse at P. Note that this ellipse is tangent to the curve [7], since the normal to the curve bisects the angle between the focal radii of the ellipse and hence is also the normal to the ellipse.

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When $\frac{1}{\rho}=\frac{1}{2}\left(\frac{1}{d_1}+\frac{1}{d_2}\right)\cos\theta$, the ellipse and the curve [7] have the same curvature at P, and the test for extremes is inconclusive. But here the conditions for a maximum or a minimum are obvious from geometrical considerations. For, remembering that d_1+d_2 is constant for points on the ellipse, say $d_1+d_2=k$, then $d_1+d_2<k$ for points *within* the ellipse and $d_1+d_2>k$ for points *without* the ellipse. *Hence the path of the ray will be a maximum or a minimum according as the curve* [7] *lies within or without the ellipse in the neighborhood of the point P; and it is seen that* [9] *and* [8] *are but special cases of this general condition.*

PROBLEM II. *Refraction at the surface* $F(x, y, z)=0$. *Using the previous notation, it is required to find the conditions that make the time of passage from* P_1 *to* P_2, *that is,* $\frac{d_1}{v_1}+\frac{d_2}{v_2}$, *an extreme, where* v_1 *and* v_2 *represent the velocity of light in the two media.*

The Lagrangian function is (§ 89)

$$\phi(x, y, z)=\frac{d_1}{v_1}+\frac{d_2}{v_2}+\lambda F(x, y, z).$$

Proceeding as in the case of reflection, we find in place of equations [3] above

[1]
$$\begin{cases}\frac{l_1}{v_1}+\frac{l_2}{v_2}=kl,\\ \frac{m_1}{v_1}+\frac{m_2}{v_2}=km,\\ \frac{n_1}{v_1}+\frac{n_2}{v_2}=kn.\end{cases}$$

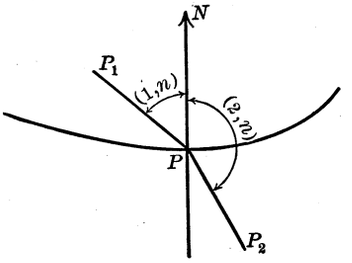

FIG. 19

From these equations we deduce that

[2]
$$\frac{1}{v_1}+\frac{\cos(1, 2)}{v_2}=k\cos(1, n),$$

[3]
$$\frac{\cos(1, 2)}{v_1}+\frac{1}{v_2}=k\cos(2, n),$$

[4]
$$\frac{\cos(1, n)}{v_1}+\frac{\cos(2, n)}{v_2}=k.$$

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Multiplying [2] by $\frac{1}{v_1}$ and [3] by $\frac{1}{v_2}$, and subtracting, we have

$$\frac{1}{v_1^2}-\frac{1}{v_2^2}=k\left[\frac{\cos(1,\,n)}{v_1}-\frac{\cos(2,\,n)}{v_2}\right];$$

substituting from [4] the value of k in this equation, it is seen that

$$\frac{1}{v_1^2}-\frac{1}{v_2^2}=\frac{\cos^2(1,\,n)}{v_1^2}-\frac{\cos^2(2,\,n)}{v_2^2},$$

or

$$\frac{\sin^2(1,\,n)}{v_1^2}=\frac{\sin^2(2,\,n)}{v_2^2}.$$

It follows that

[5]
$$\frac{\sin(1,\,n)}{v_1}=\frac{\sin(2,\,n)}{v_2}.$$

From [2] and [4] it is seen that

$$\frac{1}{v_1}+\frac{\cos(1,\,2)}{v_2}=\frac{\cos^2(1,\,n)}{v_1}+\frac{\cos(1,\,n)\cos(2,\,n)}{v_2},$$

or

$$\frac{\sin^2(1,\,n)}{v_1}=\frac{\cos(1,\,n)\cos(2,\,n)-\cos(1,\,2)}{v_2}.$$

Dividing this equation by [5] and then multiplying the result by $\sin(2,\,n)$, we find

$$\sin(1,\,n)\sin(2,\,n)=\cos(1,\,n)\cos(2,\,n)-\cos(1,\,2),$$

or

$$\cos(1,\,2)=\cos[(1,\,n)+(2,\,n)],$$

and therefore

[6]
$$(1,\,2)=(1,\,n)+(2,\,n),$$

so that the incident and the refracted ray lie in a normal plane.

Equation [5] may be put in the form

$$\frac{\sin(1,\,n)}{\sin(2,\,n)}=\frac{v_1}{v_2}=c,$$

where c is the index of refraction of the second medium with respect to the first medium. The above is a generalization of a problem due to Fermat.

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The geometrical criteria for a maximum or a minimum involves a certain Cartesian Oval whose foci are at P_1 and P_2 and which passes through P. Its equation in bipolar coördinates is

$$v_2 d_1 + v_1 d_2 = \text{const.},$$

d_1 and d_2 being the variable radii vectores. For points *on* this oval $\frac{d_1}{v_1} + \frac{d_2}{v_2}$ is a constant, say k; for points *within* this oval $\frac{d_1}{v_1} + \frac{d_2}{v_2} < k$ and for points *without* this oval $\frac{d_1}{v_1} + \frac{d_2}{v_2} > k$.

Hence the time occupied by the ray in passing from P_1 to P is a maximum or a minimum according as the curve cut from the surface by the normal plane through P_1 and P_2 lies within or without this Cartesian Oval in the neighborhood of the point P.

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CHAPTER VII

SPECIAL CASES

I. THE PRACTICAL APPLICATION OF THE CRITERIA THAT HAVE BEEN HITHERTO GIVEN AND A METHOD FOUNDED UPON THE THEORY OF FUNCTIONS, WHICH OFTEN RENDERS UNNECESSARY THESE CRITERIA

93. The practical application of the established criteria is in many cases connected with very great, if not insurmountable difficulties, which, however, cannot be disregarded in the theory. For often the solutions of the equations § 89, [c], cannot be effected without great labor, if at all, and therefore also the formation of the function ϕ is impossible. It also happens, even if the function ϕ can be formed, that the discussion regarding the coefficients of $\Delta e = 0$ is attended with much difficulty. Moreover, the formation of the function ϕ and the investigation relative to the coefficients of Δe are very often unnecessary, since through direct observation we may in many cases determine whether a maximum or a minimum really exists. If it then happens that the equations [c] admit of only one real solution (i.e. of a real system of values $x_1, x_2, \cdots, x_n$), we may be sure that this is in reality the maximum or the minimum of the function. In the same way, if we can convince ourselves a priori that both a maximum and a minimum exist, and if it happens that the equations [c] offer only two real systems of values, it is evident that the one system must correspond to the maximum value of the function, the other system to the minimum value.

The determination as to which of the two systems of values gives the one or the other is in most cases easily determined.

One cannot be too careful in the investigation whether on a position which has been determined from the equations [a] and

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[c] of § 89 there really is a maximum or a minimum, since there are cases in which one may convince himself of the existence of a maximum or a minimum, when in reality there is no maximum or minimum.

For example, to establish Euclid's theorem respecting parallel lines, one tries to prove the theorem regarding the sum of the angles of a triangle without the help of the theorem of the parallel lines. Legendre was able, indeed, to show that this sum could not be greater than two right angles; however, he did not show that they could not be less than two right angles. The method of reasoning employed at that time was as follows: If in a triangle the sum of the three angles cannot be greater than 180°, then there must be a triangle for which the maximum of the sum of these angles is really reached. Assuming this to be correct, it may be shown that in this triangle the sum of the angles is equal to 180°, and from this it may be proved that the same is true of all triangles.

We see at once that a fallacy has been made. For if we apply the same conclusions to the spherical triangles, in the case of which the sum of the angles cannot be smaller then 180°, we would find that in every spherical triangle the sum of the angles is equal to 180°, which is not true.

The fallacy consists in the assumption of the *existence* of a maximum or a minimum; it is not always necessary that an upper or a lower limit be reached, even if one can come just as near to it as is wished (see § 8).

On this account the assumption of the existence of a real maximum is not allowed without further proof. We therefore endeavor to give the *existence-proof*. For this purpose we must recall several theorems in the theory of functions.*

94. We call the collectivity of all systems of values which n variable quantities $x_1, x_2, \cdots, x_n$ can assume the *realm* (*Gebiet*) of these quantities, and each single system of values a *position* in this realm. If these quantities are variables without restriction, so that each of them can go from $-\infty$ to $+\infty$, we call the

*Note especially § 137.

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realm considered as a whole (*Gesamtgebiet*) an *n-ple multiplicity* (*n-fache Mannigfaltigkeit*). If $x_1, x_2, \cdots, x_n$ are independent of one another, then we say a definite position $(a_1, a_2, \cdots, a_n)$ lies *on the interior of the realm* if these positions and also all their neighboring positions belong to this region; it lies *upon the boundary of the realm* if in each neighborhood as small as we wish of this position there are present positions which belong to the realm, and also those that do not belong to it; it lies finally *without the defined realm* if in no neighborhood as small as we wish of this position there are positions which belong to the defined region.

If the quantities $x_1, x_2, \cdots, x_n$ are subjected to m equations of condition, then we may express these in terms of $n - m$ independent variables $u_1, u_2, \cdots, u_{n-m}$, and the same definition may be applied to these variables.

95. The following theorems are proved in the theory of functions: (1)* If a continuous variable quantity is defined in any manner, this quantity has an *upper* and a *lower* limit; that is, there is a definitely determined quantity g of such a kind that no value of the variable can be greater than g, although there is a value of the variable which can come as near to g as we wish. In the same way there is a quite determined quantity k of such a nature that no value of the variable is less than k, although there is a value of the variable that comes as near to k as we wish (see also § 8).

(2)† In the region of n variables $x_1, x_2, \cdots, x_n$, suppose we have an infinite number of positions defined in any manner; let these be denoted by $(x_1', x_2', \cdots, x_n')$. Furthermore, suppose that among the positions we have such positions that x_n' can come as near to a fixed limit a_n as we wish. Then we have in the region of the quantities $x_1, x_2, \cdots, x_n$ always at least one definite position $(a_1, a_2, \cdots, a_n)$ of such a nature that among the definite positions $(x_1', x_2', \cdots, x_n')$ there are always present positions that

* Dini, *Theorie der Functionen*, p. 68. See also a paper by Stolz, "B. Bolzano's Bedeutung in der Geschichte der Infinitesimal Rechnung," *Math. Ann.*, Vol. XVIII.

† Biermann, *Theorie der An. Funk.*, p. 81; Serret, *Calc. diff. et int.*, p. 26.

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lie as near this position as we wish; so that, therefore, if δ denotes a quantity arbitrarily small,

$$|x'_\lambda - a_\lambda| < \delta \quad (\lambda = 1, 2, \cdots, n).$$

This position lies either within or upon the boundary of the defined region $(x'_1, x'_2, \cdots, x'_n)$.

96. This presupposed, let us consider a continuous function $F(x_1, x_2, \cdots, x_n)$, and let the realm of the quantities $x_1, x_2, \cdots, x_n$ be a limited one, so that, therefore, we have systems of values which do not belong to it. If for every possible system of values $(x_1, x_2, \cdots, x_n)$ we associate the corresponding value of the function, which may be denoted by x_{n+1}, then we have defined certain positions in the region of $n+1$ quantities. For the quantity x_{n+1} there is according to the first theorem an upper limit a_{n+1}; consequently, owing to the second theorem there must be within the interior or upon the limits of the defined region a position $(a_1, a_2, \cdots, a_n, a_{n+1})$ of such a nature that in the neighborhood of this position there certainly exist positions which belong to the region in question.

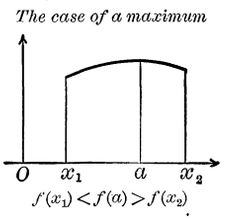

FIG. 20

Now if it can be shown that this position lies within the interior of the region, then there is in reality a maximum of the function on the position $(a_1, a_2, \cdots, a_n)$; on the contrary, if the position lies on the boundary, we cannot come to a conclusion regarding the existence of a maximum of the function x_{n+1}.

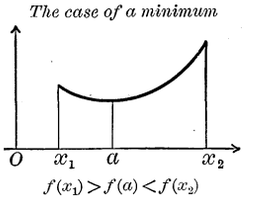

FIG. 21

It may in many cases happen that one can show, if $(x_1, x_2, \cdots, x_n)$ is any position on the boundary of the realm and if x_{n+1} denotes the corresponding value of the function, that there are present within the realm positions for which the values of the function are greater than for every position on the boundary. Then the position which we are considering here cannot lie upon the boundary, and it is clear that the limiting value of the function

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can be assumed for a definite position within the interior, since the function varies in a continuous manner. The analogue is, of course, true for a minimum. If, however, it does not admit of proof that there are positions on the interior of the defined realm for which the value of the function is greater or smaller than it is for all positions on the boundary, then nothing can be concluded regarding the real existence of a maximum or a minimum; the position $(a_1, a_2, \cdots, a_n)$ would then lie on the boundary of the region, and there might be an asymptotic approach to the limiting value a_{n+1} without this value's being in reality reached. Such cases need especial attention.

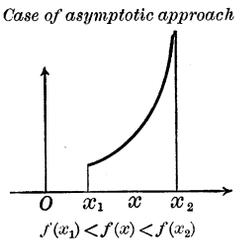

FIG. 22

The figures give a plain picture of what has been said for the case $y = f(x)$, where x is limited to the interval $(x_1 \cdots x \cdots x_2)$.

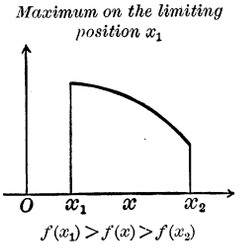

FIG. 23

97. Analogous considerations of the above are fundamental in the very definition of an analytic function. *For consider a power-series of x assumed or given in any manner; let x' be a definite value of x. Then there are three possibilities:*

(1) *x' may lie in the region of convergence of the given series or of a series that is derived* (§ 138) *from the given series; the value for $x = x'$ of this series is a value of the analytic function which is determined through the original series. In other words, if with Weierstrass we call the original series as well as any other series derived from this one with regard to the function which it represents a* ***function-element*** (***Functionenelement***), *then the first possibility consists in that, if any* ***function-element*** *is given, the definite value x' lies in the region of convergence of a function-element which is derived from the given one. We admit here also the complex variable.*

(2) *It may happen that x' does not lie in the region of convergence of any series that has been derived in this manner and*

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that we cannot derive from the original function-element another function-element whose region of convergence can come as near to the point x' as we wish. In this case the function does not exist for $x = x'$.

(3) *Although we cannot find a power-series within which x' lies, nevertheless, it sometimes happens that we may still derive elements whose regions of convergence contain positions which can come as near to the point x' as we wish.* Whether we can then define the function for $x = x'$ by the consideration of boundary conditions must in each case be considered for itself.

If we have case (1) before us, then the function is defined not only for every value x' but also for all values in the neighborhood of x' and has for these values the character of an integral function.

The definition of an analytic function as thus given is preferable to other definitions from the fact that the existence of general analytic functions is at once recognized; in short, that we have under our control, in our possession, all possible analytic functions. Every possible power-series within a region of convergence gives rise to the existence of a definite analytic function. Moreover, one must assume the duty of proving in the case of every example that it leads to just such functions.

For this reason investigations are necessary of which formerly we find no trace. If we have a differential equation, we must begin with the proof that the functions which satisfy the differential equation arise from such function-elements as we have just explained; that is, we must first show, if y is the unknown function and x is the variable of the differential equation, that this equation can be satisfied through $y = P(x - a)$. Reciprocally, if any variable quantity y is so connected with another variable quantity x that it satisfies the differential equation, we must show that it may be derived from one single function-element in the manner indicated. This last proof is of especial importance in the application of analysis to geometrical mechanics.

When a problem is given in mechanics, we have to represent the coördinates of the moving point as functions of the time.

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Only real values are permitted in this problem. We cannot therefore a priori know whether the required functions are analytic or not.

These functions are generally defined through differential equations. We shall give the simplest case as an example. Suppose we have a system of points that attract one another according to an analytic law, and let $x_1, x_2, \cdots, x_n$ be the coördinates of these points. If the motion is a free one, we have the differential equation in the form

$$\frac{d^2x_1}{dt^2} + \cdots = F(x_1, x_2, \cdots, x_n),$$

where F denotes a given function of $x_1, x_2, \cdots, x_n$. With such a problem we have to prove before everything else that the required functions of time are analytic functions. If for the point $t = t_0$ the initial position and the initial velocity are given, then in the neighborhood of the initial position we can find power-series, and we have to show that through these power-series the required functions are completely determined.

II. EXAMPLES OF IMPROPER EXTREMES WHERE THE DIFFERENCE $F(a_1+h_1, a_2+h_2, \cdots, a_n+h_n) - F(a_1, a_2, \cdots, a_n)$ IS NEITHER POSITIVE NOR NEGATIVE BUT ZERO ON THE POSITION $(a_1, a_2, \cdots, a_n)$ WHICH IS TO BE INVESTIGATED

98. We shall now consider a case which is not included in the previous investigations, but may be in a certain measure reduced to them: The definition of the proper extremes of a function consists in the fact that the difference

$$F(a_1+h_1, a_2+h_2, \cdots, a_n+h_n) - F(a_1, a_2, \cdots, a_n) \qquad (i)$$

must be invariably negative or invariably positive. There are cases where an extreme does not appear on the position $(a_1, a_2, \cdots, a_n)$ in the sense that the above difference must be positive or negative, but in the sense that the difference must be zero.

Suppose, for example, we have the problem: Determine a polygon of n sides with a given constant perimeter S whose area

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is a maximum, — a problem which we shall later discuss more fully (see § 101).

If this maximum is attained for a definite polygon, then we may at pleasure change the system of coördinates by sliding the polygon in the plane without altering the area.

For example, let $n = 3$, and (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the coördinates of the vertices of the triangle. Then the expression which is to be a maximum is

$$F = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3),$$

where the variables are subjected to the condition

$$S = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} + \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}.$$

There will not only be one system of values which gives for F a maximum value, but an infinite number of such positions; since, if we take the triangle in a definite position, we may move it in its plane at pleasure. This is therefore a case where the difference (*i*) is not positive or negative but zero.

99. Such cases, however, may be reduced to maxima and minima proper if we choose arbitrarily some of the variable quantities. In the special example of the preceding section we may assume a vertex of the triangle at pleasure; let it be the origin of coördinates, and we further assume that one of the sides coincides with the positive direction of the X-axis, so that we may write $x_1 = y_1 = y_2 = 0$. If we agree that the triangle is to lie above or below the X-axis, the problem is completely determinate.

In so far as the necessary conditions for the existence of an extreme are concerned we may proceed in precisely the same manner as we have hitherto done, since under the assumption that there are no equations of condition we have

$$F(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) - F(a_1, a_2, \cdots, a_n) = \sum_{\alpha=1}^{\alpha=n} h_\alpha F_\alpha(a_1, a_2, \cdots, a_n) + (h_1, h_2, \cdots, h_n)^{(2)}. \qquad (ii)$$

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If a minimum is to be present, then this difference can never be negative, but may be zero. For this to be possible the first derivatives must all vanish. Since, if the sum $\sum_{\alpha=1}^{\alpha=n} h_\alpha F_\alpha(a_1, a_2, \cdots, a_n)$ had (say) a positive value for $h_1 = c_1, h_2 = c_2, \cdots, h_n = c_n$, then we could place h_α equal to $c_\alpha h$ and then choose h so small that the sign of the right-hand side of (*ii*) would depend only upon the sign of the first term. If we then make h positive or negative the difference would also be positive or negative.

If equations of condition are present, it may be shown, as above, that the derivatives of the first order must vanish, since, if all these derivatives did not vanish, we might express some of the h's through the remaining ones, and then proceed as we have just done. The required systems of values $(x_1, x_2, \cdots, x_n)$ will therefore be determined from the same equations as before.

100. If we have found a system of values of the x's which satisfy the equations of condition of the problem, then in the neighborhood of this position there will be an infinite number of other positions which satisfy the equations. These last are characterized by the condition that the difference (*i*) vanishes identically for them.

This is just the condition that made impossible the former criteria, by means of which we could decide whether an extreme really entered on a position $(a_1, a_2, \cdots, a_n)$ that was determined through the equations in $x_1, x_2, \cdots, x_n$.

One must therefore seek in another manner to convince himself which case is the one in question.

This is further discussed in the following problem:

101. PROBLEM. *Among all polygons which have a given number of sides and a given perimeter, find the one which contains the greatest surface-area.* (Zenodorus.)

We see at once that the problem proposed here is of a somewhat different nature from the problems of §§ 90 and 91, since the existence of the maximum value of the function is no longer the question, as was proposed in § 49 and held as fixed throughout the general discussions. For if the definition of the maximum

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is such that the function on the position $(a_1, a_2, \cdots, a_n)$ must have a greater value on this position than on all neighboring positions, then in this sense our polygon could certainly not have a maximum area; since, if we had such a polygon on any position, we might slide the polygon at pleasure without changing its shape and consequently its area. Therefore only a maximum of the area can enter, in the sense that the periphery remaining the same an increase in the area of the surface cannot enter for an indefinitely small sliding of the end-points. We consequently cannot apply our general theory without further restriction.

102. Let the coördinates of the n end-points taken in a definite order be

$$x_1, y_1; x_2, y_2; \cdots; x_n, y_n.$$

The double area of a triangle which has the origin as one of its vertices and the coördinates of the other two vertices x_1, y_1 and x_2, y_2 is, neglecting the sign, determined through the expression

$$x_1 y_2 - x_2 y_1.$$

To determine the sign of this expression we suppose that the fundamental system of coördinates is brought through turning about its origin into such a position that the positive X-axis coincides with the length 01. We call that side of the line 01 *positive* on which lies the positive direction of the Y-axis: The double area of the triangle 012 is to be counted positive or negative according as it lies on the positive or negative side of the line 01.

If the point 0 has the coördinates x_0, y_0, the double area of the triangle is

$$2\,\Delta_{012} = (x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(x_2 - x_0),$$

where the above criterion with reference to the sign is to be applied.

For the polygon we shall take a definite consecutive arrangement of the points $(1, 2, \cdots, n)$ and, besides, we shall assume that no two of the sides cross each other. The last hypothesis is justifiable, since we may easily convince ourselves that if two sides cut each other we may at once construct a polygon whose sides do not cut one another and which, having the same perimeter as the first polygon, incloses a greater area.

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Within the polygon take a point $0 = (x_0, y_0)$ and draw from it in any direction a straight line to infinity. This straight line always cuts an odd number of sides of the polygon.

Now if we follow the periphery of the polygon in the fixed direction $(1, 2, \cdots, n)$ and mark the intersection of a side by the straight line with $+1$ or -1, according as we pass from the negative to the positive side of that line or vice versa, then the sum of these marks is either $+1$ or -1. In the first case we say that the polygon has been described in the positive direction, in the second case in the negative direction.

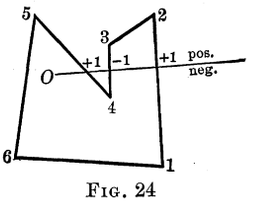

FIG. 24

It may be proved* that whatever point be taken as the point 0 within the polygon and in whatever direction the straight line be drawn, we always have the same characteristic number $+1$ or -1 if in each case the positive side of the straight line has been correctly determined.

103. The double area of the polygon is

$$2F=(x_1-x_0)(y_2-y_0)-(x_2-x_0)(y_1-y_0)+(x_2-x_0)(y_3-y_0)$$
$$-(x_3-x_0)(y_2-y_0)+\cdots+(x_n-x_0)(y_1-y_0)-(x_1-x_0)(y_n-y_0);$$

or
$$2F=x_1y_2-x_2y_1+x_2y_3-x_3y_2+\cdots+x_ny_1-x_1y_n, \qquad (\alpha)$$

where the positive or negative sign is to be taken according as the polygon has been described in the positive or negative direction. We may, however, eventually bring it about through reverting the order of the sequence of the end-points that the expression $2F$ is always positive.

104. Suppose that this has been done. The function $2F$ is to be made a maximum under the condition that the periphery has a definite value S.

We may write

$$s_{1,2}+s_{2,3}+\cdots+s_{n-1,n}+s_{n,1}=S, \qquad (\beta)$$

where
$$s_{\lambda-1,\lambda}=\sqrt{(x_\lambda-x_{\lambda-1})^2+(y_\lambda-y_{\lambda-1})^2}. \qquad (\gamma)$$

* The proof is found in Cremona, *Elementi di geometria projetiva*. Rome, 1873.

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Form the function

$$G = 2F + e(s_{1,2} + s_{2,3} + \cdots + s_{n,1} - S), \qquad (\delta)$$

and placing its partial derivatives equal to 0, we have

$$\left.\begin{aligned} \frac{\partial G}{\partial x_\lambda} &= y_{\lambda+1} - y_{\lambda-1} + e\left(\frac{x_\lambda - x_{\lambda+1}}{s_{\lambda+1,\lambda}} + \frac{x_\lambda - x_{\lambda-1}}{s_{\lambda-1,\lambda}}\right) = 0 \\ \frac{\partial G}{\partial y_\lambda} &= -x_{\lambda+1} + x_{\lambda-1} + e\left(\frac{y_\lambda - y_{\lambda+1}}{s_{\lambda+1,\lambda}} + \frac{y_\lambda - y_{\lambda-1}}{s_{\lambda-1,\lambda}}\right) = 0 \end{aligned}\right\} \qquad (\epsilon)$$

($\lambda = 1, 2, \cdots, n$; however, for $\lambda = n$ we must write $\lambda + 1 = 1$).

Take in addition the equation (β) and we have $2n+1$ equations for the determination of the $2n+1$ unknown quantities

$$x_1, x_2, \cdots, x_n, \quad y_1, y_2, \cdots, y_n, \quad e.$$

105. To reach in the simplest manner the desired result from these equations, we adopt the following mode of procedure. If we write

$$z_\lambda = (x_\lambda - x_{\lambda-1}) + i(y_\lambda - y_{\lambda-1}), \qquad (\zeta)$$

then z_λ, geometrically interpreted, represents the length from the point $\lambda - 1$ to the point λ both in value and in direction.

If, further, we write

$$z'_\lambda = (x_\lambda - x_{\lambda-1}) - i(y_\lambda - y_{\lambda-1}), \qquad (\eta)$$

then

$$z_\lambda \cdot z'_\lambda = s^2_{\lambda-1,\lambda}. \qquad (\theta)$$

Multiply the first of equations (ϵ) by i and subtract from the result the second; then owing to (ζ) we have

$$z_\lambda + z_{\lambda+1} + ei\left(\frac{z_\lambda}{s_{\lambda-1,\lambda}} - \frac{z_{\lambda+1}}{s_{\lambda,\lambda+1}}\right) = 0,$$

or

$$\left.\begin{aligned} z_\lambda\left(1 + \frac{ei}{s_{\lambda-1,\lambda}}\right) &= -z_{\lambda+1}\left(1 - \frac{ie}{s_{\lambda,\lambda+1}}\right) \\ z'_\lambda\left(1 - \frac{ei}{s_{\lambda-1,\lambda}}\right) &= -z'_{\lambda+1}\left(1 + \frac{ei}{s_{\lambda,\lambda+1}}\right) \end{aligned}\right\}. \qquad (\iota)$$

Now, multiplying the last two equations together, we have from (θ)

$$s^2_{\lambda-1,\lambda}\left(1 + \frac{e^2}{s^2_{\lambda-1,\lambda}}\right) = s^2_{\lambda,\lambda+1}\left(1 + \frac{e^2}{s^2_{\lambda,\lambda+1}}\right),$$

and therefore

$$s^2_{\lambda-1,\lambda} = s^2_{\lambda,\lambda+1}.$$

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106. Since $s_{\nu-1,\nu}$ is an essentially positive quantity, it follows that

$$s_{\lambda-1,\lambda} = s_{\lambda,\lambda+1}, \qquad (\kappa)$$

and consequently the sides of the polygon are all equal to one another. Hence each side $= \frac{S}{n}$, and we have from (ι)

$$\frac{z_{\lambda+1}}{z_\lambda} = \frac{ein + S}{ein - S} = \text{const.}$$

If we write

$$z_\lambda = \frac{S}{n} e^{\phi\lambda i},$$

where ϕ_λ denotes the angle which $s_{\lambda-1,\lambda}$ makes with the X-axis, then

$$e^{(\phi_{\lambda+1} - \phi_\lambda)i} = \text{const.},$$

or

$$\phi_{\lambda+1} - \phi_\lambda = \text{const.}; \qquad (\lambda)$$

that is, all the angles of the polygon are equal to one another, and consequently the polygon is a regular one.

It is thus shown that the conditions which are had from the vanishing of the first derivatives can be satisfied only by a regular polygon; that is, if there is a polygon which, with a given perimeter and a prescribed number of sides, has a greatest area, this polygon is necessarily regular.

Our deductions, however, have in no manner revealed that a maximum really exists.

107. To establish the existence of a maximum we must apply the method given in §§ 93–96. We note that an upper limit exists for the area of the polygon, from the fact that the number of sides and the perimeter are given; for if we consider a square whose sides are greater than the given perimeter S, we can lay each polygon with the perimeter S in this square, and in such a way that the end-points of the polygon do not fall upon the sides of the square. Hence the area of the polygon cannot be greater than that of the square, and consequently there must be an upper limit for this area, which may be denoted by F_0. The question is, Can this limit in reality be reached for a definite system of values? The variables $x_1, y_1; x_2, y_2; \cdots; x_n, y_n$ being limited to this square, there must be (§ 96) among the positions $(x_1, y_1; x_2, y_2; \cdots; x_n, y_n)$ which fill out the square a position

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$(a_1, b_1; a_2, b_2; \cdots; a_n, b_n)$ of such a nature that in every neighborhood of this position other positions exist for which the corresponding surface area F of the polygon formed from them comes as near as we wish to the upper limit. We may assume that this position is within the square, since if it lies by chance on the boundary, then from what has been said above, it is admissible to slide the corresponding polygon without altering its shape and area into the interior of the square.

We assert that the value of the function F for the position $(a_1, b_1; a_2, b_2; \cdots; a_n, b_n)$ must necessarily be equal to F_0. For if this was not the case, the inequality must also remain if we subject the points $a_1, b_1; a_2, b_2; \cdots; a_n, b_n$ to an indefinitely small variation; and on account of the continuity of F it would not be possible in the arbitrary neighborhood of $(a_1, b_1; \cdots; a_n, b_n)$ to give positions for which the corresponding area comes arbitrarily near the upper limit F_0. This, however, contradicts the conclusions previously made. Hence all n corners with a given periphery not only approach a definite limit with respect to their inclosed area but this limit is in reality reached. Since, furthermore, the necessary conditions for the existence of a maximum have given the regular polygon of n sides as the only solution, and since we have seen a maximum really exists, we may with all rigor make the conclusion: *That polygon which, with a given periphery and a given number of sides, contains the greatest area is the regular polygon.*

PROBLEM

Among the regular polygons with a constant periphery, the one with the greatest number of angles has the greatest area. (Zenodorus.)

108. Hadamard's problem. If $A_1=(x_1, y_1, z_1)$, $A_2=(x_2, y_2, z_2)$, $A_3=(x_3, y_3, z_3)$ are the rectangular coördinates of any three points from a fixed origin O, the volume formed on the three lines OA_1, OA_2, OA_3 is

$$\Delta=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix};$$

and if

$$x_i^2+y_i^2+z_i^2=d_i^2 \quad (i=1, 2, 3),$$

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where d_1, d_2, d_3 are positive constants, it may be easily shown that Δ is a maximum when $\Delta = d_1 \cdot d_2 \cdot d_3$; or, *of all parallelopipedons constructed on the three sides OA_1, OA_2, OA_3, the one having the greatest volume is the rectangular parallelopipedon.* As the parallelopipedon may occupy an infinite number of positions without changing the origin, we have here a case of improper maximum which is of interest.

The extension of this problem is due to Hadamard.*

$$If\ \Delta = \begin{vmatrix} x_{11}, & x_{12}, & \cdots, & x_{1n} \\ x_{21}, & x_{22}, & \cdots, & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1}, & x_{n2}, & \cdots, & x_{nn} \end{vmatrix},$$

where $x_{i1}^2 + x_{i2}^2 + \cdots + x_{in}^2 = d_i^2$ $(i = 1, 2, \cdots, n)$, *the d's being positive constants, show that the maximum of the absolute value of* Δ *is*

$$\Delta = d_1 \cdot d_2 \cdot \ \cdots \ \cdot d_n.$$

This may be done as follows:

Let the determinant be developed with respect to the elements of the ith line, so that

$$\Delta = A_{i1}x_{i1} + A_{i2}x_{i2} + \cdots + A_{in}x_{in}. \qquad (i)$$

We then have to find the maximum or the minimum of the function Δ of the n variables $x_{i1}, x_{i2}, \cdots, x_{in}$ which are connected by the relation

$$x_{i1}^2 + x_{i2}^2 + \cdots + x_{in}^2 = d_i^2. \qquad (ii)$$

The Lagrange method (§ 89) leads at once to the conditions

$$\frac{x_{i1}}{A_{i1}} = \frac{x_{i2}}{A_{i2}} = \cdots = \frac{x_{in}}{A_{in}}. \qquad (iii)$$

If $x_{k1}, x_{k2}, \cdots, x_{kn}$ are the elements of another line of the determinant, we have

$$A_{i1}x_{k1} + A_{i2}x_{k2} + \cdots + A_{in}x_{kn} = 0; \qquad (iv)$$

or, from (ii),

$$x_{i1}x_{k1} + x_{i2}x_{k2} + \cdots + x_{in}x_{kn} = 0, \qquad (v)$$

where

$$i \neq k.$$

* Hadamard (*Bull. des Sciences Mathématiques*, Second Series, Vol. XVII, 1893). Proof by Wirtinger (ibid., 1908). An interesting application of this problem is found in Bôcher, *Introduction to the Study of Integral Equations*, pp. 31 et seq.

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From this we conclude that *the determinant can only have an extreme value when it is orthogonal.*

When the conditions (v) exist, the square of the determinant is another determinant, in which all the elements are zero except those of the principal diagonal, which are $d_1^2, d_2^2, \cdots, d_n^2$.

It follows that $$\Delta^2 = d_1^2 \cdot d_2^2, \cdots, d_n^2.$$

Here again we have an improper extreme which it is interesting to consider further.

III. CASES IN WHICH THE SUBSIDIARY CONDITIONS ARE NOT TO BE REGARDED AS EQUATIONS BUT AS LIMITATIONS

109. Besides the problems already mentioned, those problems are particularly deserving of notice in which the conditions for the variables are not given in the form of equations but as restrictions or limitations.

For example, let a point in space and a function which depends upon the coördinates of this point be given. Furthermore, let the point be so restricted that it always remains within the interior of an ellipsoid; then the restriction made upon the point is expressed through the inequality $$0 \leqq \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1.$$

We have, accordingly, such limitations when a function of the variables is given which cannot exceed a certain upper and a certain lower limit.

We make such a restriction when we assume that a function f_1 shall always lie between fixed limits a and b.

110. This limitation, which consists of two inequalities

$$[\alpha] \qquad a < f_1 < b,$$

may be easily reduced to one.

For from $[\alpha]$ it follows necessarily that

$$[\beta] \qquad \frac{f_1 - a}{b - f_1} > 0;$$

and, reciprocally, if $[\beta]$ exists and if $a < b$, then f_1 must be situated between a and b and consequently $[\alpha]$ must be true.

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Every limitation of the kind given may be analytically represented as one single inequality of the form $[\beta]$.

111. We must next find the algorithm for the cases under consideration. This may be done at once if we consider that such cases may be reduced to those in which occur equations of condition. For this purpose we need only establish the problem of finding the maximal or minimal values of a function whose variables are subjected to certain conditions as follows:

It is required among all systems of values which satisfy the equations $f_{\lambda=0}(\lambda = 1, 2, \cdots, m)$ *to find those for which* F *is a maximum or a minimum.*

By proposing the problem in this manner, it is clear that all the variables x which appear in the equations of condition need not necessarily be contained in the function.

Suppose further we have the limitation that

$$[\gamma] \qquad f_k > 0,$$

then, through the introduction of a new variable x_{n+1}, we may transform this limitation into an equation of condition. For, as we have to do with only real values of the variables, the equation

$$[\gamma^\alpha] \qquad f_k = x_{n+1}^2$$

denotes exactly the same thing as $[\gamma]$.

If, therefore, a function $F(x_1, x_2, \cdots, x_n)$ is to be a maximum or minimum under the limitations

$$f_1 = 0, f_2 = 0, \cdots, f_m = 0, f_{m+1} > 0, f_{m+2} > 0, \cdots, f_{m+r} > 0,$$

where the f's are functions of $x_1, x_2, \cdots, x_n$, then we may solve this problem if instead of the r last restrictions we introduce the following limitations:

$$f_{m+1} = x_{n+1}^2, f_{m+2} = x_{n+2}^2, \cdots, f_{m+r} = x_{n+r}^2.$$

The problem is thus reduced to the one of finding among the systems of variables $x_1, x_2, \cdots, x_{n+r}$ those systems for which F is a maximum or a minimum.

112. Examples of this character occur very frequently in mechanics. As an example consider *a pendulum which consists*

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of a flexible thread that cannot be stretched. The condition under which the motion takes place is not that the material point remains at a constant distance from the origin, but that the distance cannot be greater than the length of the thread. Such problems are more closely considered in the sequel. It will be seen that by means of ***Gauss's principle*** *all problems of mechanics may be reduced to problems of maxima and minima.*

IV. GAUSS'S PRINCIPLE

113. For the sake of what follows we shall give a short account of this principle: Consider the motion of a system of points whose masses are $m_1, m_2, \cdots, m_n$. Let the motions of the points be limited or restricted in any manner, and suppose that the system moves under the influence of forces that act continuously. For a definite time let the positions of the points and the components of velocity both in direction and magnitude be determined. The manner in which the motion takes place from this period on is determined through Gauss's principle:

Let $A_1, A_2, \cdots, A_n$ be the positions of the points at the moment first considered; $B_1, B_2, \cdots, B_n$ the positions which the points can take after the lapse of an indefinitely small time τ, if the motions of these points are free; $C_1, C_2, \cdots, C_n$ the positions in which these points *really* are after the lapse of the same time τ; and, finally, let $C_1', C_2', \cdots, C_n'$ be the positions which the points may also possibly have assumed after the time τ, when the conditions are fulfilled.

If we form

$$\sum_{\nu=1}^{\nu=n} m_\nu \overline{B_\nu C_\nu}^2 \quad \text{and} \quad \sum_{\nu=1}^{\nu=n} m_\nu \overline{B_\nu C_\nu'}^2,$$

it follows from Gauss's principle that from $\tau = 0$ up to a definite value of τ the condition

$$[1] \qquad \sum_{\nu=1}^{\nu=n} m_\nu \overline{B_\nu C_\nu}^2 < \sum_{\nu=1}^{\nu=n} m_\nu \overline{B_\nu C_\nu'}^2$$

is always satisfied; that is, $\sum_{\nu=1}^{\nu=n} m_\nu \overline{B_\nu C_\nu}^2$ must always be a minimum.

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114. To make rigorous deductions from Gauss's principle, which was briefly sketched in the preceding section, we shall give a more analytic formulation of it: For this purpose we denote the coördinates of A_ν by x_ν, y_ν, z_ν, the components of the velocity of A_ν by x_ν', y_ν', z_ν', and the components of the force acting upon A_ν by X_ν, Y_ν, Z_ν. The coördinates of B_ν are therefore

$$x_\nu + \tau x_\nu' + \frac{\tau^2}{2} X_\nu,\ y_\nu + \tau y_\nu' + \frac{\tau^2}{2} Y_\nu,\ z_\nu + \tau z_\nu' + \frac{\tau^2}{2} Z_\nu;$$

and from Taylor's theorem the coördinates of C_ν are

$$x_\nu + \tau x_\nu' + \frac{\tau^2}{2} x_\nu'' + \cdots,\ y_\nu + \tau y_\nu + \frac{\tau^2}{2} y_\nu^2 + \cdots,\ z_\nu + \tau z_\nu' + \frac{\tau^2}{2} z_\nu'' + \cdots;$$

consequently we have

$$[2]\quad \sum m_\nu \overline{B_\nu C_\nu}^2 = \sum m_\nu \{(x_\nu'' - X_\nu)^2 + (y_\nu'' - Y_\nu)^2 + (z_\nu'' - Z_\nu)^2\} \frac{\tau^4}{4} + \cdots.$$

Instead of x_ν'', however (see preceding section), other values may possibly enter, say $x_\nu'' + \xi_\nu, \cdots$, so that we have

$$[3]\quad \sum_{\nu=1}^{\nu=n} m_\nu \overline{B_\nu C_\nu'}^2 = \sum_{\nu=1}^{\nu=n} m_\nu \{(x_\nu'' + \xi_\nu - X_\nu)^2 + (y_\nu'' + \eta_\nu - Y_\nu)^2 + (z_\nu'' + \zeta_\nu - Z_\nu)^2\} \frac{\tau^4}{4} + \cdots.$$

It follows from Gauss's principle that the difference of the sums [2] and [3] must always be positive.

Hence

$$[4]\quad 0 > \sum_{\nu=1}^{\nu=n} m_\nu \{2\,[\xi_\nu (x_\nu'' - X_\nu) + \eta_\nu (y_\nu'' - Y_\nu) + \zeta_\nu (z_\nu'' - Z_\nu)] + \xi_\nu^2 + \eta_\nu^2 + \zeta_\nu^2\} \frac{\tau^4}{4} + \cdots;$$

that is, the quantities x_ν'', y_ν'', z_ν'' must be such that the sum [2] is a minimum.

Hence, among all the x_ν'', y_ν'', z_ν'' which are associated with the conditions of motion, we must seek those which make [2] a minimum.

115. We have reached our proposed object if we can show that the ordinary equations of mechanics may be derived from Gauss's principle.

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If there are no equations of condition present, then clearly [2] is only a minimum when

$$x_\nu'' = X_\nu,\ y_\nu'' = Y_\nu,\ z_\nu'' = Z_\nu.$$

If, however, we have equations of condition, for example, if any of the variables are connected by a relation such as $f(x, y, z) = 0$, then these must hold true throughout the whole motion. They may therefore be differentiated. We have in this way equations in $\frac{dx_\nu}{dt}$, $\frac{dy_\nu}{dt}$, and $\frac{dz_\nu}{dt}$. Differentiate again and we have equations in x_ν'', y_ν'', and z_ν''.

Hence, in conformity with the rules that have been hitherto found for the theory of maxima and minima, the quantities x_ν'', y_ν'', z_ν'' are to be so determined that the derived equations of condition are satisfied, while at the same time [2] becomes a minimum. But in this case also, as is easily shown, we are led to the usual differential equations of mechanics.

116. The theory of maxima and minima may be applied to realms which are seemingly distant from it. An example in question is the proof of a very important theorem in the theory of functions.

The Reversion of Series

If the following n equations exist among the variables $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$,

$$[1] \qquad \begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + \mathbf{X}_1, \\ y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \mathbf{X}_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + \mathbf{X}_n, \end{cases}$$

where the coefficients on the right-hand side are given finite quantities and the $\mathbf{X}$*'s are power-series in the x's of such a nature that each single term is higher than the first dimension, and if the series on the right-hand side are convergent and the*

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determinant of the nth order of the linear functions of the x's which appear in [1], *namely,*

$$[2] \qquad A=\begin{vmatrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22}, & \cdots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \cdots, & a_{nn} \end{vmatrix},$$

is different from zero, then, reciprocally, the x's may also be expressed through convergent series of the n quantities y which identically satisfy the equations [1].

117. As an algorithm for the representation of the series for the x's, we make use of the following methods (cf. §§ 135, 136):

If we solve the equations [1] linearly by bringing the terms of the higher powers of the x's on the left-hand side, we have

$$x_1=\frac{A_{11}}{A}(y_1-\mathrm{X}_1)+A_{12}(y_2-\mathrm{X}_2)+\cdots+\frac{A_{1n}}{A}(y_n-\mathrm{X}_n),$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$x_n=\frac{A_{n1}}{A}(y_1-\mathrm{X}_1)+\frac{A_{n2}}{A}(y_2-\mathrm{X}_2)+\cdots+\frac{A_{nn}}{A}(y_n-\mathrm{X}_n),$$

where $A_{\lambda\mu}$ denotes the corresponding first-minor of $a_{\lambda\mu}$ in [2].

It is seen that in general

$$[3] \quad x_\lambda=\sum_{\mu=1}^{\mu=n}\frac{A_{\lambda\mu}}{A}(y_\mu-\mathrm{X}_\mu)=\sum_{\mu=1}^{\mu=n}\frac{A_{\lambda\mu}}{A}y_\mu-\frac{[x_1, x_2, \cdots, x_n]_\lambda^{(2)}}{A},$$

where $[x_1, x_2, \cdots, x_n]_\lambda^{(2)}$ denotes terms of the second and higher dimensions in $x_1, x_2, \cdots, x_n$.

We shall therefore have a first approximation to the result if we consider only the terms on the right-hand side of [3] which are of the first dimension. A second approximation is reached if we substitute in the right-hand side of [3] the first approximations already found and reduce everything to terms of the second dimension inclusive. Continuing with the second approximations that have been found, substitute them in [3] and, neglecting all terms above the third dimensions, we have the third approximation, etc.; we may thus derive the x's to any degree of exactness required.

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Since A is found in all the denominators, the development converges the more rapidly the greater A is.

118. In what follows we shall assume that the quantities on the right-hand side of [1] are all real and that we may write

$$\mathbf{X}_\lambda = H_{\lambda 2} + H_{\lambda 3} + H_{\lambda 4} + \cdots \quad (\lambda = 1, 2, \cdots, n),$$

where $H_{\lambda i}$ is a homogeneous function of the ith degree in x_1, $x_2, \cdots, x_n$, and consequently by Euler's theorem for homogeneous functions

$$\begin{aligned}\mathbf{X}_\lambda = {} & \frac{1}{2} x_1 \frac{\partial H_{\lambda 2}}{\partial x_1} + \frac{1}{2} x_2 \frac{\partial H_{\lambda 2}}{\partial x_2} + \cdots + \frac{1}{2} x_n \frac{\partial H_{\lambda 2}}{\partial x_n} \\ & + \frac{1}{3} x_1 \frac{\partial H_{\lambda 3}}{\partial x_1} + \frac{1}{3} x_2 \frac{\partial H_{\lambda 3}}{\partial x_2} + \cdots + \frac{1}{3} x_n \frac{\partial H_{\lambda 3}}{\partial x_n} \\ & + \frac{1}{4} x_1 \frac{\partial H_{\lambda 4}}{\partial x_1} + \frac{1}{4} x_2 \frac{\partial H_{\lambda 4}}{\partial x_2} + \cdots + \frac{1}{4} x_n \frac{\partial H_{\lambda 4}}{\partial x_n} \\ & + \cdots\cdots\cdots\cdots\cdots \\ & \qquad (\lambda = 1, 2, \cdots n);\end{aligned}$$

or,
$$\mathbf{X}_\lambda = x_1 \mathbf{X}_{\lambda 1} + x_2 \mathbf{X}_{\lambda 2} + \cdots + x_n \mathbf{X}_{\lambda n},$$

where the quantities $\mathbf{X}_{\lambda\mu}$ $(\lambda = 1, 2, \cdots, n;\ \mu = 1, 2, \cdots, n)$ are continuous functions of the x's, which become indefinitely small with the x's.

The system of equations [1] may then be brought to the form

$$[1a] \qquad y_\lambda = \sum_{\mu=1}^{\mu=n} (a_{\lambda\mu} + \mathbf{X}_{\lambda\mu})\, x_\mu \quad (\lambda = 1, 2, \cdots, n).$$

The theorem of § 116 in this modified form may be expressed as follows:

(1) *It is always possible so to fix for the variables $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$, limits $g_1, g_2, \cdots, g_n$ and $h_1, h_2, \cdots, h_n$ that for every system of the y's for which $|y_\lambda| < h_\lambda$ there exists* **one** system of the x's for which $|x_\lambda| < g_\lambda$, and in such a way that the equations [1a] are satisfied.*

* See Biermann, *Theo. der An., Funk.*, p. 234, and also Stolz, p. 172.

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(2) *The solution of the equations* $[1a]$ *has a form similar to the equations* $[1a]$ *themselves, viz.:*

$$[3a] \qquad x_\lambda = \sum_{\mu=1}^{\mu=n} \left(\frac{A_{\lambda\mu}}{A} + \mathbf{Y}_{\lambda\mu} \right) y_\mu \quad (\lambda = 1, 2, \cdots, n),$$

where the $\mathbf{Y}_{\lambda\mu}$ *are continuous functions of the y's, which become indefinitely small with these quantities.*

To prove this theorem we make use of the theory of maxima and minima.

119. If we give to the y_λ the value zero, the equations $[1a]$ can only be satisfied if their determinant vanishes, that is, when

$$[4] \quad \left| a_{\lambda\mu} + \mathbf{X}_{\lambda\mu} \right| = 0 \quad (\lambda = 1, 2, \cdots, n;\ \mu = 1, 2, \cdots, n),$$

except for the case where the x's vanish.

For sufficiently small values of the x's the determinant [4] is not very different from the determinant [2]. We may therefore determine limits g for the x's so that [4] cannot be zero unless [2] is also zero. $A = 0$ is, however, by hypothesis excluded. Accordingly the y's can only be zero in $[1a]$ when all the x's vanish, provided the x's are confined *within* fixed limits. These limits may be regarded as the *boundaries* of a definite realm.

120. Again, we write

$$[5] \qquad y_\lambda = \sum_{\mu=1}^{\mu=n} \left(a_{\lambda\mu} + \mathbf{X}_{\lambda\mu} \right) x_\mu = F_\lambda(x_1, x_2, \cdots, x_n)$$
$$(\lambda = 1, 2, \cdots, n),$$

and consider the function

$$[6] \qquad S = \sum_{\lambda=1}^{\lambda=n} F_\lambda^2 (x_1, x_2, \cdots, x_n).$$

In S we shall write for the x's all the systems of values where at least one x lies on the boundary of the realm in question. The realm of the x's is thus the totality of the x's for which

$$\left| a_{\lambda\mu} + \mathbf{X}_{\lambda\mu} \right| \text{ is zero only when } \left| a_{\lambda\mu} \right| = 0$$
$$(\lambda = 1, 2, \cdots, n;\ \mu = 1, 2, \cdots, n);$$

it follows then that [4] is not zero, since $\left| a_{\lambda\mu} \right|$ is by hypothesis different from zero.

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When one of the x's reaches its limit, there is no system of values of the x's for which the function [6] vanishes, since the function can (as follows from definition [5] of the F_λ and the considerations of § 119) only vanish if all the y's and consequently all the x's vanish.

There is then a lower limit G which is different from zero for the values of [6] which correspond to a definite system of values $(x_1, x_2, \cdots, x_n)$ of the boundaries.

121. We come next to the determination of the limiting values of the y's. For this purpose we consider the expression

$$[7] \qquad \sum_{\lambda=1}^{\lambda=n}\left(F_\lambda(x_1, x_2, \cdots, x_n) - y_\lambda\right)^2.$$

If we ascribe definite values to the y's, then there is for the values [7] in the realm of the x's a system for which [7] is a minimum.

We wish to show that this system of values of the x's does not lie upon the boundary of the realm. We prove this by showing that there is a point *within* the realm where the expression [7] has a smaller value than it has on the boundary.

The expression [7] may be written

$$\sum_{\lambda=1}^{\lambda=n}(F_\lambda - y_\lambda)^2 = S - 2\sum_{\lambda=1}^{\lambda=n}F_\lambda y_\lambda + \sum_{\lambda=1}^{\lambda=n}y_\lambda^2.$$

Since $\sqrt{S}$ is at all events greater than F_λ, and consequently

$$\frac{F_\lambda}{\sqrt{S}} < 1,$$

it follows that

$$\sum_{\mu=1}^{\mu=n}F_\mu y_\mu < \sqrt{S}\sum_{\mu=1}^{\mu=n}y_\mu < \sqrt{S}\left|\sum_{\mu=1}^{\mu=n}y_\mu\right| < \sqrt{S}\sum_{\mu=1}^{\mu=n}|y_\mu| < \sqrt{S}\sum_{\mu=1}^{\mu=n}h_\mu,$$

where the h's are the limits of the y's. From this it results that

$$\sum_{\mu=1}^{\mu=n}(F_\mu - y_\mu)^2 > S - 2\sqrt{S}\sum_{\mu=1}^{\mu=n}h_\mu + \sum_{\mu=1}^{\mu=n}y_\mu^2,$$

and, consequently, for a greater reason

$$[8] \qquad \sum_{\mu=1}^{\mu=n}(F_\mu - y_\mu)^2 > S - 2\sqrt{S}\sum_{\mu=1}^{\mu=n}h_\mu.$$

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The limits h_λ must be so chosen that the right-hand side of [8] is positive. This choice can be made so that the expression on the right-hand side for a system of x's which belongs to the boundary does not become arbitrarily small but always remains greater than a certain lower limit (see the preceding section).

The expression, however, on the interior of the realm of the x's may be arbitrarily small, viz., when $x_1 = x_2 = \cdots = x_n = 0$.

For this system of values the left-hand side of [8] is equal to

$$\sum_{\mu=1}^{\mu=n} y_\mu^2.$$

We have therefore found the following result: *We can give limits g to the variables x, and to the y's the limits h, in such a way that the expression* [7] *for systems of values of the x's which belong to the boundary of the realm is always greater than it is for the zero position* ($x_1 = x_2 = \cdots = x_n = 0$).

Hence the position for which the expression [7] *is a minimum must necessarily lie within the realm of the x's; and we may be certain that within the realm of the x's there is a position where* [7] *has its smallest value.*

122. In order to find the minimal position of [7] which was shown to exist in the previous section we must differentiate the function [7] and put the first partial derivatives equal to 0.

This gives

$$[9] \qquad \sum_{\nu=1}^{\nu=n} (F_\nu - y_\nu) \frac{\partial F_\nu}{\partial x_\mu} = 0 \quad (\mu = 1, 2, \cdots, n).$$

These n equations can, in case the determinant

$$[10] \qquad \left| \frac{\partial F_\nu}{\partial x_\mu} \right| \quad (\nu = 1, 2, \cdots, n;\ \mu = 1, 2, \cdots, n)$$

is different from zero, exist only if the quantities within the brackets vanish.

[10] is identical with the determinant [4]; and (see § 119) it may be always brought about through suitable choice of the limits g of the x's that [4] is different from zero if only the determinant [2], as by hypothesis is the case, is different from zero.

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Hence the equation [9] can only be satisfied if

[1a] or [5] $$y_\nu = F_\nu(x_1, x_2, \cdots, x_n).$$

We have therefore found that, *since there is certainly a system of values of the x's for which the function $\sum_{\nu=1}^{\nu=n}(y_\nu - F_\nu)^2$ is a minimum, there must also be a system within the realm of the x's for which the equations* [1a] *are satisfied if to the y's definite values in their realm are arbitrarily given.*

123. We must further see whether within the fixed realm there is one or several systems of values of the x's that satisfy the equations [1a] with prescribed values of the y's which lie within definite limits.

To establish this, assume that $(x_1', x_2', \cdots, x_n')$ is a second system of values that satisfy the equations [1a]; we must then have the equations

[11] $$F_\nu(x_1', x_2', \cdots, x_n') - F_\nu(x_1, x_2, \cdots, x_n) = 0 \quad (\nu = 1, 2, \cdots, n).$$

Developing by Taylor's theorem, we have, when we consider only terms of the first dimension,

[11a] $$\sum_{\rho=1}^{\rho=n}(x_\rho' - x_\rho)\left\{\frac{\partial F_\nu}{\partial x_\rho} + \overline{\mathrm{X}}_{\nu,\rho}\right\} = 0 \quad (\nu = 1, 2, \cdots, n).$$

The $\overline{\mathrm{X}}_{\nu,\rho}$ are functions which depend upon the x''s and x's and vanish with these quantities.

We may determine the n unknown quantities $x_1' - x_1$, $x_2' - x_2$, $\cdots$, $x_n' - x_n$ from the n linear equations [11a].

For small values of x and x' the determinant

[12] $$\left|\frac{\partial F_\nu}{\partial x_\rho} + \overline{\mathrm{X}}_{\rho,\nu}\right| \quad (\nu, \rho = 1, 2, \cdots, n)$$

will be little different from the determinant [10].

We may therefore make the limits g of the x's so small that [12] is different from zero for all the x's and x''s which belong to the realm; and when this has been done, the equations [11a] are only satisfied for $$x_\rho' = x_\rho \quad (\rho = 1, 2, \cdots, n);$$

that is, *there exists within the realm in question no second system of the x's which satisfies the equations* [1a].

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We have therefore come to the following result:

It is possible so to determine the limits g and h that with every arbitrary system of the y's in which each single variable does not exceed its definite limiting value, the given equations [1 *a*] *are satisfied by* **one** *and* **only one** *system of the x's in which these quantities likewise do not exceed their limits.**

The first part of the theorem given in § 118 is thus proved.

REMARK. We have assumed that we have to do only with real quantities. The discussion, however, is not restricted to such quantities, as it is easy to prove that the same developments may also be made for complex variables.

124. The values of the x's which were had from the equations

$$[1\,a] \qquad y_\nu = \sum_{\rho=1}^{\rho=n} (a_{\nu\rho} + \mathbf{X}_{\nu\rho}) x_\rho \quad (\nu = 1, 2, \cdots, n)$$

may be derived in the manner given in § 118.

If we write

$$\left| a_{\nu\rho} + \mathbf{X}_{\nu\rho} \right| = A' \quad (\nu = 1, 2, \cdots, n;\ \rho = 1, 2, \cdots, n),$$

the linear solution of the equations [1 *a*] is

$$[3\,b] \qquad x_\rho = \sum_{\nu=1}^{\nu=n} \frac{A'_{\nu\rho}}{A'} y_\nu \quad (\rho = 1, 2, \cdots, n),$$

where $A'_{\nu\rho}$ denotes the corresponding first minors. Now A' is a definite quantity which lies within certain finite limits; the same is also true of $\frac{1}{A'}$. $A'_{\nu\rho}$ is found in a similar manner. Hence the quantities $\frac{A'_{\nu\rho}}{A'}$ are finite quantities which lie between definite limits; and, therefore, if the y's become indefinitely small, the x's will also become indefinitely small; that is, those systems of values of the x's which satisfy the equations [1] under the named conditions are, as has also been shown in § 119, so formed that they become indefinitely small with the y's.

* See also Hadamard, "Sur les transformations ponctuelles," *Bull. de la Société Math.*, Vol. XXXIV, 1906.

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We may now show that the x's are continuous functions of the y's.

Let $(b_1, b_2, \cdots, b_n)$ be a definite system of values of the y's, and let the system $(a_1, a_2, \cdots, a_n)$ of the x's correspond to this system of the y's.

If we then write

$$[13] \qquad \left.\begin{aligned} y_\lambda &= b_\lambda + \eta_\lambda \\ x_\lambda &= a_\lambda + \xi_\lambda \end{aligned}\right\} \quad (\lambda = 1, 2, \cdots, n),$$

the system of equations $[1\,a]$ or $[1]$ goes into

$$\eta_\lambda + b_\lambda = F_\lambda(a_1 + \xi_1, a_2 + \xi_2, \cdots, a_n + \xi_n) \quad (\lambda = 1, 2, \cdots, n),$$

$$[1\,b] \quad \eta_\lambda = F_\lambda(a_1 + \xi_1, a_2 + \xi_2, \cdots, a_n + \xi_n) - F_\lambda(a_1, a_2, \cdots, a_n)$$
$$(\lambda = 1, 2, \cdots, n].$$

Developing this expression according to powers of the ξ's, we have

$$[1\,c] \qquad \eta_\lambda = \sum_{\mu=1}^{\mu=n} C'_{\lambda\mu}\xi_\mu \quad (\lambda = 1, 2, \cdots, n),$$

where the $C'_{\lambda\mu}$ are functions of the a's and ξ's. If the ξ's are indefinitely small, we may limit the $C'_{\lambda\mu}$ to the first derivatives of F_λ. In this case we denote the coefficients of $[1\,c]$ by $C_{\lambda\mu}$, so that

$$C_{\lambda\mu} = \frac{\partial F_\lambda}{\partial x_\mu} \text{ for } (x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n)$$
$$(\lambda = 1, 2, \cdots, n\,;\ \mu = 1, 2, \cdots, n),$$

and the determinant of the equations $[1\,c]$ goes into

$$\left|\frac{\partial F_\lambda}{\partial x_\mu}\right| \text{ for } (x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n)$$
$$(\lambda = 1, 2, \cdots, n\,;\ \mu = 1, 2, \cdots, n).$$

If the x's lie within definite limits, this determinant remains always above a definite limit. We may therefore say that the determinant has a value different from zero. Consequently the condition that the equations $[1\,c]$ may be solved is satisfied, and it is seen that indefinitely small values of the ξ's must correspond to indefinitely small values of the η's.

This means nothing more than that the functions x are continuous functions of the y's.

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125. The above investigations are true under the assumption that the functions F_λ are continuous, that their first derivatives exist and likewise are continuous within certain limits. We need know absolutely nothing about the second derivatives.

Of the x's, of which it is already known that they exist as functions of the y's and vary in a continuous manner with them, it may now likewise be proved that they, considered as functions of the y's, have derivatives which are continuous functions of the y's.

The proof in question may be derived from the following considerations: If from [1c] we express the ξ's in terms of the η's, we have

$$[3c] \qquad \xi_\mu = \sum_{\lambda=1}^{\lambda=n} \frac{\Delta_{\lambda,\mu}}{\Delta} \eta_\lambda \quad (\mu = 1, 2, \cdots, n).$$

The $\frac{\Delta_{\lambda\mu}}{\Delta}$ are continuous functions of the ξ's, and the ξ's are continuous functions of the η's. Hence $\frac{\Delta_{\lambda\mu}}{\Delta}$ may be represented as continuous functions of the η's.

If the η's become indefinitely small, then the ξ's become indefinitely small, and we have definite limits for $\frac{\Delta_{\lambda\mu}}{\Delta}$.

In general, if we have a function $f(x_1, \cdots, x_n)$ of the n variables $x_1, x_2, \cdots, x_n$, and if we consider the difference

$$f(a_1 + \xi_1, a_2 + \xi_2, \cdots, a_n + \xi_n) - f(a_1, a_2, \cdots, a_n),$$

it is seen that it may be written in the form

$$\sum_{\lambda=1}^{\lambda=n} (b_\lambda + H_\lambda)\, \xi_\lambda,$$

where the H_λ depend upon the ξ's and become indefinitely small with these quantities, and the b_λ are, in virtue of the definition of the differential quotient, the partial differential quotients of f with respect to x_λ for the system of values $(a_1, a_2, \cdots, a_n)$. From the above it results not only that the x's are continuous functions of the y's but also that the derivatives of the first order of these functions exist.

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We have, indeed, the derivatives of the first order if in the expressions $\frac{\Delta_{\lambda\mu}}{\Delta}$ we write the ξ's equal to zero.

The quantities $\frac{\Delta_{\lambda\mu}}{\Delta}$, however, become then, in accordance with $[1c]$, the quantities which we should have in $[1c]$ if we had at first written $C_{\lambda\mu}$ instead of $C'_{\lambda\mu}$.

But the quantities $C_{\lambda\mu}$ are continuous functions of $a_1, a_2, \cdots, a_n$. We may therefore say that the differential quotients $\frac{\Delta_{\lambda\mu}}{\Delta}$ are continuous functions of the variables x, and it is then proved that the x's are such functions of the y's as the y's are of the x's.

126. For the complete solution of the second part of the theorem in § 116 we have yet to show that the expressions $[3b]$ may be reduced to the form $[3a]$.

For this purpose we must bring the quantities $\frac{A'_{\lambda\mu}}{A'}$ in $[3b]$ (§ 124) to the form

$$\frac{A'_{\lambda\mu}}{A'} = b_{\lambda\mu} + \mathbf{Y}_{\lambda\mu},$$

where $b_{\lambda\mu}$ is the value of $\frac{A'_{\lambda\mu}}{A'}$ when all the x's are equal to zero. $\mathbf{Y}_{\lambda\mu}$ is a function of the x's, but the x's are functions of the y's, so that $\mathbf{Y}_{\lambda\mu}$ is a function of the y's which vanishes when they vanish.

We may therefore in reality write $[3b]$ in the form $[3a]$

$$[3a] \qquad x_\mu = \sum_{\lambda=1}^{\lambda=n} (b_{\lambda\mu} + \mathbf{Y}_{\lambda\mu}) y_\mu \quad (\mu = 1, 2, \cdots, n).$$

127. There may arise cases in which we know nothing further of the functions F_λ, as was assumed in § 116, than that they are real continuous functions.

We cannot then conclude, for example, that the x's may be developed in powers of the y's; but we may reduce the equations to the form $[3a]$ and show that the equations $[1a]$ are solvable.

The theorem which has been proved is of great importance when applied to special cases, even for elementary investigations.

If, for example, the equation $f(x, y) = 0$ is given, then it is taught in the differential calculus how we can find the derivative of y considered as a function of x.

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If we assume that the variables x and y are limited to a special realm where the two derivatives with respect to x and y do not vanish and therefore the curve $f(x, y) = 0$ has no singular points, and if the equation is satisfied by the system (x_0, y_0), we may write $x = x_0 + \xi$, $y = y_0 + \eta$. We have then $f(x_0 + \xi, y_0 + \eta) = 0$, and we may prove with the aid of the theorem in § 118 that η is a continuous function of ξ and has a first derivative. Not before this has been done have we a right to differentiate and proceed according to the ordinary rules of the differential calculus.

MISCELLANEOUS PROBLEMS

1. Show that the problem of determining the extremes of the function $f(x, y)$ may be reduced to the determination of the upper and lower limits of this function under the condition that $x^2 + y^2 = r^2$. (Stolz, *Wiener Ber.*, Vol. C, p. 1167.)

2. Find the shortest distance from the point $P(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$.

Answer. $\dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$.

3. Find the points on a given sphere which are the farthest from and nearest to a given plane which does not intersect the sphere. (Pappus.)

4. Find the triangle of maximum area whose vertices V_1, V_2, and V_3 describe respectively three given plane curves C_1, C_2, and C_3. When the three curves reduce to the same ellipse, show that there are an infinity of triangles of maximum area (a case of improper maximum).

5. Find the ellipse of least area that may be drawn through the three vertices of a triangle.

6. Find the ellipsoid of least volume which may be drawn through the four vertices of a tetrahedron.

7. In a triangle of greatest or least area circumscribed about a curve, the points of contact are the mid-points of the sides.

8. Among the triangles whose vertices are situated respectively upon three given straight lines in space, which is the one whose perimeter gives a maximum or minimum? Also determine the triangle of maximum or minimum area.

Answer. In the first case the bisectrices of the triangle are respectively normal to the straight lines described by the vertices; in the second case the altitudes of the triangle are perpendicular to these lines.

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9. Upon a fixed surface find a point P such that the sum of the squares of its distances from n fixed points $A_1, A_2, \cdots, A_n$ is a maximum or a minimum.

Answer. If the tangent plane at P is taken as the xy-plane and the normal to this plane at P as the z-axis, the center of mean distances M, say, of the points A lie upon the z-axis. It follows that the points P are the feet of the normals which may be drawn to the surface from M.

10. Show that the semi-axes of a central section of a quadric

$$A_1x^2 + A_2y^2 + A_3z^2 + 2B_1yz + 2B_2zx + 2B_3xy + 1 = 0$$

are the roots r^2 of the equation

$$\begin{vmatrix} 1 + A_1r^2, & B_3, & B_2, & l \\ B_3, & 1 + A_2r^2, & B_1, & m \\ B_2, & B_1, & 1 + A_3r^2, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0,$$

where the section is made by the plane

$$lx + my + nz = 0.$$

11. Show that the axes of the quadric of the preceding example are the roots of the following cubic in r^2:

$$\begin{vmatrix} 1 + A_1r^2, & B_3r^2, & B_2r^2 \\ B_3r^2, & 1 + A_2r^2, & B_1r^2 \\ B_2r^2, & B_1r^2, & 1 + A_3r^2 \end{vmatrix} = 0.$$

12. $z = \frac{1}{2}v - y$ is to be a maximum, where $y^3 - nyx + x^3 = 0$ and $v - x = y$. (Hudde, 1658. See Descartes, *Geom.*, Vol. I, pp. 507–516.)

13. *The fundamental theorem of algebra.* Let $f(t)$ be an integral function of t with constant coefficients. Write $t = x + iy$, so that

$$f(t) = P(x, y) + iQ(x, y) = P + iQ, \tag{1}$$

with the identical relations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \quad \text{and} \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}. \tag{2}$$

Form the expression $\mu(x, y) = \mu = P^2 + Q^2$. Within the circle of radius $r = \sqrt{x^2 + y^2}$ the function μ is everywhere continuous, so that (§ 8) the function μ must reach its lower limit for values of x and y within or on the boundary of the circle. By taking r sufficiently large it is seen that the lower limit of μ must be reached *within* the circle, so that there must be a *minimum* value of μ. Show that this minimum value is zero, and consequently that there must be some value of t which makes $f(t) = 0$, provided that $f(t)$ is *not* a constant. In particular the semi-definite case must be considered.

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CHAPTER VIII

CERTAIN FUNDAMENTAL CONCEPTIONS IN THE THEORY OF ANALYTIC FUNCTIONS

I. ANALYTIC DEPENDENCE; ALGEBRAIC DEPENDENCE

128. If in the development of the conception of the analytic functions we start with the simplest functions which may be expressed through arithmetical operations, we come first to the *rational* functions of one or more variables.* The conception of these rational functions may be easily extended by substituting in their places *one-valued* functions, and first of all those which may be expressed through arithmetical operations, viz., sums of an infinite number of terms of which each is a rational function, or products of an infinite number of such functions.

Such a transcendental function is, for example,

$$v(x) = u_1(x) + u_2(x) + \cdots + u_k(x) + u_{k+1}(x) + \cdots,$$

where $u_i(x)\,[i = 1, 2, \cdots]$ are rational functions of x. The necessity at once arises of developing the conditions of convergence of infinite series and products, since such an arithmetical expression represents a definite function only for values of the variable for which it converges. Mere convergence, however, is not sufficient if we wish to retain for the functions just mentioned the properties which belong to the rational and the ordinary transcendental functions. All such functions have derivatives, and we shall restrict ourselves once for all to functions which have derivatives.

Furthermore, the derived series of the above expressions of one variable must converge *uniformly (gleichmässig)* in the neighborhood of each definite value, and every term of the derived series

* See Hancock, *Elliptic Functions*, Vol. I, pp. 6-9.

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must be continuous in the same neighborhood. (Osgood, *Lehrbuch der Functionentheorie*, p. 83.)

129. When we say that a series whose terms are functions of one variable *converges uniformly*, we mean the following:* It is assumed that the series in question has a definite value for $x = x_0$. We consider all values of x for which $x - x_0$ does not exceed a definite quantity d. This determines a fixed region for x, within which we shall suppose that the series is convergent. This region is known as the *region of convergence* (*Convergenzbezirk*). We may, for brevity, put

$$R_k(x) \text{ for } u_{k+1} + u_{k+2} + \cdots$$

in the series above. In order that this series converge uniformly, it must be possible after we have assumed an arbitrarily small positive quantity δ, and when a remainder $R_k(x)$ has been separated from the series, to find a positive integer m so that

$$|R_k(x)| \leqq \delta, \text{ where } k > m$$

for all values of x within the region of convergence.†

130. Proceeding in this way we may form more complicated expressions; for example, we may let $w(x)$ be a sum of an infinite number of terms where each term is a transcendental function like $v(x)$ above, so that

$$w(x) = v_1(x) + v_2(x) + \cdots.$$

We may continue by forming similar expressions out of the transcendental functions $w(x)$, etc. It is clear that if we proceed in this manner, there is no end of such expressions, so that even if we limit ourselves to one-valued functions, we do not obtain a clear insight into the possible kinds and forms of such functions.

It is essential that all such transcendental functions have a common property, and we note that if we take a value x_0 within the region of convergence in which the series representing these

* Weierstrass, *Collected Works*, Vol. II, p. 202, and *Zur Functionenlehre*, § 1.

† See Dini, *Theorie der Funktionen* (page 137 of the German translation by Lüroth and Schepp).

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functions converge uniformly, they may be represented for all the values of x in the neighborhood of x_0 as series which proceed according to positive integral powers of $x - x_0$; for example, in the form

$$F(x) = F(x - x_0 + x_0) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots,$$

where $a_0, a_1, a_2, \cdots$, are definite functions of x_0. From this it follows that they may be differentiated, and a number of other properties are immediate consequences.

131. We may next extend the conception of uniform convergence to functions of several variables. With Weierstrass (loc. cit.) consider the infinite series

$$F(x_1, x_2, \cdots, x_n) = \sum_{\nu=0}^{\nu=\infty} u_\nu(x_1, x_2, \cdots, x_n)$$

whose terms u_ν are functions of an arbitrary number of variable quantities $x_1, x_2, \cdots, x_n$. Such a function *converges uniformly* in any part (R) of its region of convergence when with a prescribed quantity δ chosen arbitrarily small there exists a positive integer m such that the absolute value of

$$\sum_{\nu=k}^{\nu=\infty} u_\nu(x_1, x_2, \cdots, x_n) \leqq \delta$$

for every value of k which is $\geqq m$ and for every system of values of $x_1, x_2, \cdots, x_n$ which belongs to (R).

Let $a_1, a_2, \cdots, a_n$ be a definite system of values of the variables $x_1, x_2, \cdots, x_n$ within the region of uniform convergence, and consider only the values of $x_1, x_2, \cdots, x_n$ for which $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$ do not exceed certain limits $d_1, d_2, \cdots, d_n$, as in § 129.

The function may then be represented through an ordinary series which proceeds according to integral powers of $x_1 - a_1, x_2 - a_2, x_3 - a_3, \cdots, x_n - a_n$, and consequently may be differentiated; in short, it behaves, as Weierstrass * was accustomed to express it, like an integral rational function in the neighborhood of a definite position within the interior of the region of uniform convergence.

* In this connection see Weierstrass, *Werke*, Vol. II, pp. 135 et seq., and also Biermann, *Theorie der Analy. Funktionen*, pp. 429 et seq.

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132. We may next introduce the conception of *analytic dependence.* If we represent a function which has been formed as indicated above by $F(x_1, x_2, \cdots, x_n)$, then $F(x_1, x_2, \cdots, x_n) = 0$ expresses a certain dependence among the variables $x_1, x_2, \cdots, x_n$; that is, among the infinite number of systems of values for which the function has a meaning, those only which satisfy this equation are to be considered. There exists, therefore, among $x_1, x_2, \cdots, x_n$ a dependence of a similar character, as in the case of algebraic equations. If we choose the quantities $x_1, x_2, \cdots, x_n$ such that the equations $F_1 = 0, F_2 = 0, \cdots, F_m = 0$ exist where $m < n$, we have a dependence among the quantities $x_1, x_2, \cdots, x_n$ defined in such a way that at all events we can choose arbitrarily not more than $n - m$ of the variables, since the remaining m variables are determined.

133. The conception of the *many-valued functions* is at once suggested. Suppose, for example, a function of two variables x and y is given; then we may consider all the systems of values (x, y) in which x has a prescribed value. For such a value of x there may exist several values of y. We are to regard y as a function of x, and this function is a *many-valued* function. By the introduction of one or more auxiliary variables it is often possible to express the many-valued functions* through one-valued functions, and indeed in algebraic form. The development of analytic functions from an arithmetical or algebraic standpoint seemed especially desirable to Weierstrass. He wrote (see *Werke*, Vol. II (Oct. 3, 1875), p. 235): "The more I consider the underlying principles of the theory of functions — and I do this continually — the stronger am I convinced that this theory must be built upon the foundation of algebraic truths."

134. To illustrate the remarks of the preceding article, consider any analytic dependence existing between, say, two variables x and y and limit one of the variables x to a definite region. The other variable must be expressed through x and in a form that remains invariably true for all values of x in question. Now, if to the one variable there corresponds a transcendental function,

*We might cite, for example, the Abelian transcendents.

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it does not seem possible to express one variable *arithmetically* in terms of the other. We may, however, introduce a third auxiliary variable and thereby express both of the original variables as one-valued functions of the third variable and in such a way that, if we give to this variable all possible values, we have all systems of values of (x, y).

The simplest example is perhaps the one given by the equation $z = x^y$, where z and y are two independent variables. It is not possible to express the dependence between x and y in an arithmetical form; that is, one in which transcendentals do not appear. But if we introduce a third variable t, and write $x = e^t$, we have $z = e^{yt}$, so that $y = \frac{\log z}{t}$.

Thus x and y are expressed as one-valued functions of t, and such that for one value of x there is invariably one value of t and of y. Poincaré* proved that if x and y are connected by an algebraic equation, then all systems of values (x, y) may be expressed in the form just indicated. He also showed that if any analytic dependence exists between x and y, it is always possible to represent x and y as one-valued functions of a third variable. An example in point is the expression of the integrals of linear differential equations through the *Fuchsian functions*. However, he did not show in this latter case that *all* the points of the region in question were thus expressed through t. On the contrary it seems that there exists an infinite number of isolated points which can be reached only when t tends toward certain limits.

For example, in the differential equation of the hypergeometric series, we should have to exclude in such a representation the real values of x from $+1$ to $+\infty$. (See the Paris Thesis of Goursat.)

In this manner the study of many-valued functions may be reduced to the study of one-valued functions. However, it is not

*See *Bulletin de la Société mathématique de France*, Vol. XI (1883). See also lectures II and III, delivered in the Cambridge Colloquium, by Professor Osgood (*Bulletin of the Amer. Math. Society*, 1898); and the *Problèmes mathématiques* of Professor Hilbert in the *Comptes rendus* of the Congress of Mathematicians, Paris, 1900. In his treatment of Algebraic Functions of Two Variables, Professor Picard has done valuable work in this connection.

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a simple task, for if we wish in reality to make this representation even in the case of a linear differential equation, we encounter many technical difficulties. Nevertheless, it is essential to prove that there exist such representations.

Weierstrass asserted (May, 1884) that he believed the following theorem existed in the theory of functions: *It is always possible, where an analytic dependence exists, to express this dependence in a one-valued form which remains invariably true.*

135. We may next introduce the following theorem, which is extensively used, particularly in the calculus of variations:

Suppose that between the variables $x_1, x_2, \cdots, x_n$ *we have* m *equations given which may be represented in the form of power-series, and let these be*

$$\begin{aligned}
&c_{1,1}(x_1-a_1)+\cdots+c_{1,n}(x_n-a_n)+\mathbf{X}_1=0,\\
&c_{2,1}(x_1-a_1)+\cdots+c_{2,n}(x_n-a_n)+\mathbf{X}_2=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&c_{m,1}(x_1-a_1)+\cdots+c_{m,n}(x_n-a_n)+\mathbf{X}_m=0 \quad (m<n),
\end{aligned}$$

where $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_m$ *are also power-series of* $x_1-a_1, \cdots, x_n-a_n$, *but of such a nature that each term in them is of a higher dimension than the first.*

The equations will be satisfied for $x_1=a_1, \cdots, x_n=a_n$. *We propose the problem of determining all systems of values* $(x_1, x_2, \cdots, x_n)$ *which lie in the neighborhood of* $(a_1, a_2, \cdots, a_n)$ *and which satisfy the* m *equations above; that is, among the systems of values for which* $|x_1-a_1|, \cdots, |x_n-a_n|$ *are smaller than a fixed limit* ρ, *determine those which satisfy the* m *equations.*

The quantity ρ is subject to the condition only of being sufficiently small. To solve this problem we consider the system of linear equations to which the given equations reduce when we write

$$\mathbf{X}_1=0, \mathbf{X}_2=0, \cdots, \mathbf{X}_m=0.$$

Through these linear equations m of the differences x_1-a_1, $x_2-a_2, \cdots, x_m-a_m$ may be expressed in terms of the $n-m$

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remaining, if the determinants of the mth order which may be formed out of the m rows of the c's are not all zero.

If, say,

$$\begin{vmatrix} c_{1,1}, & \cdots, & c_{1,m} \\ \vdots & & \vdots \\ c_{m,1}, & \cdots, & c_{m,m} \end{vmatrix} \gtrless 0,$$

we have (§ 117)

$$\begin{aligned}
x_1 - a_1 &= c'_{1,1}(x_{m+1} - a_{m+1}) + \cdots + c'_{1,n-m}(x_n - a_n) + \mathbf{X}'_1, \\
x_2 - a_2 &= c'_{2,1}(x_{m+1} - a_{m+1}) + \cdots + c'_{2,n-m}(x_n - a_n) + \mathbf{X}'_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_m - a_m &= c'_{m,1}(x_{m+1} - a_{m+1}) + \cdots + c'_{m,n-m}(x_n - a_n) + \mathbf{X}'_m.
\end{aligned}$$

By means of these equations we may represent $x_1 - a_1$, $x_2 - a_2$, $\cdots$, $x_m - a_m$ as power-series in the remaining $n - m$ differences, the formal procedure being as follows:

We write $\mathbf{X}'_1 = 0, \cdots, \mathbf{X}'_m = 0$, and thus obtain for $x_1 - a_1$, $\cdots$, $x_m - a_m$ expressions which represent the first approximations. These are substituted in $\mathbf{X}'_1, \cdots, \mathbf{X}'_m$, and in the resulting expressions only terms of the second dimension are considered. These terms added to the terms of the first approximations respectively constitute the second approximations. Continuing this process we may represent the required expressions to any degree of exactness desired.

We obtain the same results if we express m of the quantities $x_1 - a_1$, $x_2 - a_2$, $\cdots$, $x_n - a_n$ through power-series in terms of the remaining $n - m$ quantities with indeterminate coefficients. These coefficients may be determined without difficulty.

As just shown, these power-series are convergent as soon as the differences $x - a$ which enter into them do not exceed certain limits, and, furthermore, these power-series satisfy the given equations.

136. The problem of the preceding article may be solved in the following more symmetric manner, in which none of the variables is given preference over the others (see Lagrange, *Théorie des Fonctions*, Vol. II, § 58).

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Besides the equations given above we introduce others which are likewise expressed in power-series:

$$\text{Let} \quad c_{m+1,1}(x_1 - a_1) + \cdots + c_{m+1,n}(x_n - a_n) + \mathbf{X}_{m+1} = t_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$c_{n,1}(x_1 - a_1) + \cdots + c_{n,n}(x_n - a_n) + \mathbf{X}_n = t_{n-m},$$

where $t_1, t_2, \cdots, t_{n-m}$ are $n - m$ new or auxiliary variables.

The quantities c are arbitrarily chosen, in such a manner, however, that the determinant

$$\begin{vmatrix} c_{1,1}, & c_{1,2}, & \cdots, & c_{1,m}, & c_{1,m+1}, & \cdots, & c_{1,n} \\ c_{2,1}, & c_{2,2}, & \cdots, & c_{2,m}, & c_{2,m+1}, & \cdots, & c_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m,1}, & c_{m,2}, & \cdots, & c_{m,m}, & c_{m,m+1}, & \cdots, & c_{m,n} \\ c_{m+1,1}, & c_{m+1,2}, & \cdots, & c_{m+1,m}, & c_{m+1,m+1}, & \cdots, & c_{m+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n,1}, & c_{n,2}, & \cdots, & c_{n,m}, & c_{n,m+1}, & \cdots, & c_{n,n} \end{vmatrix} \gtrless 0.$$

Proceeding as in § 117 we write the quantities $\mathbf{X}$ equal to zero, and we thus have a system of n linear equations through which we can express the n differences $x_1 - a_1, \cdots, x_n - a_n$ through $t_1, t_2, \cdots, t_{n-m}$, say,

$$x_\nu - a_\nu = e_{\nu\,1}t_1 + e_{\nu,2}t_2 + \cdots + e_{\nu,n-m}t_{n-m} + \mathbf{X}'_\nu$$
$$(\nu = 1, 2, \cdots, n).$$

With the help of these equations we can express $x_1 - a_1, \cdots, x_n - a_n$ as power-series in $t_1, \cdots, t_{n-m}$.

To do this we again write $\mathbf{X}'_\nu = 0$, and have only terms of the first dimension. We write the first approximations that have been thus obtained in $\mathbf{X}'_\nu$ and by retaining the terms of the second dimension derive the second approximations, etc.

It follows directly from the above that these power-series in t *formally* satisfy the given equation; that they possess a certain common region of convergence if we give certain fixed limits to $|t_1|, |t_2|, \cdots, |t_{n-m}|$; that they consequently *in reality* satisfy the equations; and, finally, that *all the systems of values* $(x_1, x_2, \cdots, x_n)$ which lie in the neighborhood of $(a_1, a_2, \cdots, a_n)$ and which satisfy the proposed equations are obtained in this way.

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137. Suppose that between two variables there exists an analytic relation which is expressed in the form

$$P(x - x_0, y - y_0) = 0,$$

where P denotes simply a power-series and where x_0, y_0 is a definite pair of values of the variables.

In the neighborhood of (x_0, y_0) there is an infinite number of systems of values which satisfy the equation. The collectivity of these pairs of values (x, y) is called an *analytic structure*, or configuration (*Gebild*), in the realm (*Gebiet*) of the quantities (x, y).

We may next make an application of the theorem of the preceding article. It follows that, if between n quantities $x_1, x_2, \cdots, x_n$ there exist m equations in the form of power-series, then the differences $x_1 - a_1, \cdots, x_m - a_m$ may be expressed through power-series of the $n - m$ remaining variables. Weierstrass said: "Through the m equations a structure of the $(n-m)$th kind in the realm of the n quantities $x_1, x_2, \cdots, x_n$ is defined."

As in the case of two variables, we may proceed in a similar manner with several variables, among which an analytic dependence exists. Let this connection be of such a nature that $m\ (<n)$ of the variables are in general determined through the remaining $n - m$. If, then, $(a_1, a_2, \cdots, a_n)$ represents a definite system of values of the variables, there exist m equations of the form

$$P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) = 0$$

which are to be satisfied for $x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n$. In the neighborhood of the position $(a_1, a_2, \cdots, a_n)$ there are, then, an infinite number of other systems of values $(x_1, x_2, \cdots, x_n)$ which satisfy the same m equations. These define an *analytic structure* in the realm of the quantities* $x_1, x_2, \cdots, x_n$.

A fundamental theorem in the theory of functions of the complex variable is that these structures may be *continued* over their boundaries. The power-series above constitutes an *element* of a complete structure (§ 97).

*Weierstrass, *Werke*, Vol. II, p. 236. It may be remarked that Minkowski in his *Geometrie der Zahlen* advances similar ideas at considerable length. See in particular § 19 of his work just mentioned.

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138. Analytical structures, as above defined, may be represented in a different manner. If the equation connecting x and y begins with terms of the first dimension, we may, on the one hand, either express $y - y_0$ through $P(x - x_0)$ or $x - x_0$ through $P(y - y_0)$; or, on the other hand, if the coefficient of either $x - x_0$ or $y - y_0$ is equal to zero, it is possible to express only $y - y_0$ or only $x - x_0$ as integral power-series of $x - x_0$ or $y - y_0$. In order that this distinction may not be necessary, we introduce a function t which begins with terms of the first dimension in $x - x_0$, $y - y_0$ (see § 136); we may then always express the two quantities x, y as power-series of t. Through the introduction of such a quantity t it is made possible to include within certain limits all the systems of values $(x - x_0, y - y_0)$ which satisfy this equation. These values must *firstly* satisfy the given equation, and *secondly* they must afford all the systems of values which satisfy it within these limits.

These considerations may be extended at once to equations in several variables. If we have a certain number of equations in $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$, and if we limit these equations to terms of the first dimension, we have linear homogeneous equations of the first dimension, the number of which we assume to be m $(< n)$.

If we can express m of the quantities $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$ through the remaining $n - m$, it is always possible so to derive n power-series of $n - m$ quantities $t_1, t_2, \cdots, t_{n-m}$ that they, substituted for $x_1, x_2, \cdots, x_n$, *firstly* satisfy the given equations, and *secondly*, if we give to $t_1, t_2, \cdots, t_{n-m}$ all possible values, they offer all the systems of values $(x_1, x_2, \cdots, x_n)$ which satisfy those equations, when certain limits are fixed for the absolute values of $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$; or, also *secondly*, that with indefinitely small values of the t's they afford all the systems of values of the quantities $x_1, x_2, \cdots, x_n$ which lie indefinitely near the position $(a_1, a_2, \cdots, a_n)$ (see again § 136).

Take n power-series $\Phi_1(t), \Phi_2(t), \cdots, \Phi_n(t)$ and write $x_1 = \Phi_1(t)$, $x_2 = \Phi_2(t), \cdots, x_n = \Phi_n(t)$; then through these equations a *structure of the first kind* (*Stufe*) *in the realm of the n quantities x*

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is defined; in a similar manner a *structure of the second kind* is defined through the equations

$$x_1 = \Phi_1(t_1, t_2),\ x_2 = \Phi_2(t_1, t_2),\ \cdots,\ x_n = \Phi_n(t_1, t_2), \text{ etc.}$$

In general, if we take n power-series in $t_1, t_2, \cdots, t_{n-m}$ and write these equal to $x_1, x_2, \cdots, x_n$, the collectivity of the systems of values $(x_1, x_2, \cdots, x_n)$ offered through these equations constitute a *structure of the* $(n-m)^{th}$ *kind in the realm of the quantities* $x_1, x_2, \cdots, x_n$.

We shall in the sequel limit the discussion of the general analytic dependence to the cases where this dependence is expressed through algebraic equations and to the structures which result from such equations, viz., the *algebraic structures.*

II. ALGEBRAIC STRUCTURES IN TWO VARIABLES

139. Let $F(x, y)$ be an integral algebraic function of x and y which does not contain repeated factors, so that $F(x, y)$ has no common factor with either $\frac{\partial F}{\partial x}$ or $\frac{\partial F}{\partial y}$. Further suppose that $F(x, y)$ is *not* divisible by any integral function in which appears only one of the variables x or y. The system of values x, y which satisfy the equation $F(x, y) = 0$ form the *algebraic structure* that is defined through this equation.

If x_0, y_0 is a pair of values such that $F(x_0, y_0) = 0$, we may develop the equation $F(x_0, y_0)$ in powers of $x - x_0$ and $y - y_0$ in the form (cf. Stolz, loc. cit., p. 177)

$$[1] \qquad G(\xi, \eta) = dF(x_0, y_0) + \frac{1}{2!} d^2F(x_0, y_0) + \cdots + \frac{1}{n!} d^nF(x_0, y_0) + \cdots = 0,$$

where for brevity we put $x - x_0 = \xi, y - y_0 = \eta$, and where $d^nF(x_0, y_0)$ is the homogeneous function of the nth degree in ξ, η, viz.,

$$[2] \qquad d^nF(x_0, y_0) = \sum_{r=0}^{r=n} \binom{n}{r} \frac{\partial^n F}{\partial x_0^{n-r} \partial y_0^r} \xi^{n-r} \eta^r.$$

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If $\frac{\partial F}{\partial x_0}$ and $\frac{\partial F}{\partial x_0}$ do not both vanish, the position (or point) x_0, y_0 is said to be *regular* or *simple.* But if they both vanish for $x = x_0$, $y = y_0$, and if for the same position all the partial derivatives of the 2d, 3d, $\cdots$, $(k-1)$st order of $F(x, y)$ with respect to x and y vanish, while those of the kth order are *not* all zero, the position x_0, y_0 is called a *singular position,* and, specifically, a k-ple singularity. In such a case the left-hand side of equation [1] begins with terms of the kth order with respect to ξ and η.

In the following treatment not only the integer k plays an important rôle but also the smallest exponent of the terms that are free from η, as also the smallest exponent of the terms that are free from ξ, on the left-hand side of [1]. If we denote the first by p and the second by q, the equation [1] may be written in the form

$$[3] \quad F(x_0+\xi, y_0+\eta) = \xi^p\{a+\xi f(\xi)\}+\eta^q\{b+\eta g(\eta)\}+\xi\eta H(\xi,\eta)=0.$$

Here $f(\xi)$ denotes an integral function of ξ and $g(\eta)$ an integral function of η; a and b are constants different from zero, viz.,

$$a=\frac{1}{p!}\frac{\partial^p F}{\partial x_0^p}, \quad b=\frac{1}{q!}\frac{\partial^q F}{\partial y_0^q}.$$

140. Developments of the algebraic function y in the neighborhood of a regular position. It may be shown * that *if on the position* $x = x_0$, $y = y_0$ *the expression* $\frac{\partial F}{\partial y}$ *does not vanish* $\left(\text{*so that, say,* } q=1 \text{ *and* } b=\frac{\partial F}{\partial y_0}\right)$, *there is one and only one convergent series in integral positive powers of* ξ *which vanishes with* ξ *and which substituted for* η *in* [1] *identically satisfies* [1].

We may suppose that this series begins with ξ^p, so that

$$[4] \quad \eta = -\frac{a}{b}\xi^p + c_{p+1}\xi^{p+1}+\cdots.$$

We have also to consider in the sequel *fractional* positive powers of $x - x_0 = \xi$; that is, powers, say, $\xi^{\frac{1}{\mu}}$, where $\mu \neq 1$. A series

* See Pierpont, Vol. I, p. 288; or Goursat, *Cours D'Analyse*, Vol. I, chap. iii.

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of this kind is *convergent* if there is a positive quantity R such that the series for all values of $|\xi^{\frac{1}{\mu}}| < R$ is convergent, and that is for all values of $|\xi| < R^{\mu}$.

If the series is convergent for one of the μ values of the μth root of ξ, it is evidently convergent for all the other $\mu - 1$ values of ξ. Accordingly, to each of the values of ξ whose absolute value is smaller than R^{μ} there correspond μ different values of the series. If, for example, we denote a definite one of the values of ξ, for example, the principal one, by $\xi^{\frac{1}{\mu}}$, the others are expressed through the product $j\xi^{\frac{1}{\mu}}$, where j is any of the μth roots of unity.

Hence a series

$$[5] \qquad \sum_{n=\lambda}^{n=\infty} c_n \left(\sqrt[\mu]{\xi}\right)^n \quad (\lambda > 0)$$

may, corresponding to the different values of j, appear in the $\mu - 1$ other forms

$$\sum_{n=\lambda}^{n=\infty} c_n j^n \left(\sqrt[\mu]{\xi}\right)^n.$$

The theorem stated at the beginning of this article may be generalized: If on the position $x = x_0$, $y = y_0$ the expression $\dfrac{\partial F}{\partial y}$ does not vanish, there is one and *only one* convergent power-series in positive integral *or fractional* powers of ξ which vanishes with ξ and which written for η in the equation [1] identically satisfies it, viz., the series [4]. For if besides the series [4] a series [5] with $\mu > 1$ satisfied [1], then the equation

$$[6] \qquad F(x_0 + t^{\mu}, y_0 + \eta) = 0, \quad \text{where } t = \xi^{\frac{1}{\mu}},$$

would be satisfied by two series which have no constant term and in which η is expressed in integral powers of t. This, by the previous theorem, is impossible, because in [6] the term in η really appears, and in fact multiplied by the coefficient $\dfrac{\partial F}{\partial y_0}$.

141. Suppose next that $\dfrac{\partial F}{\partial y_0} = 0$, but that $\dfrac{\partial F}{\partial x_0} \neq 0$, so that $p = 1$ and $q > 1$. Then from what we have just seen it follows that the

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equation [3] may be solved through only convergent series in which ξ is expressed in powers of η in the form

$$[7] \qquad \xi = -\frac{b}{a}\eta^q + d_s\eta^{q+s} + \cdots = Q(\eta), \text{ say,}$$

where $q > 1$ and $d_s \neq 0$; in other words, there exists a positive quantity S such that if $|\eta| < S$, we have the identity

$$[8] \qquad G(Q(\eta), \eta) = 0.$$

Write ξ in the form

$$\xi = -\frac{b}{a}\eta^q\left[1 - \frac{a}{b}d_s\eta^s + \cdots\right] = -\frac{b}{a}\eta^q Q_1(\eta), \text{ say,}$$

and note that

$$[Q_1(\eta)]^{\frac{1}{q}} = \left(1 - \frac{ad_s}{b}\eta^s + \cdots\right)^{\frac{1}{q}} = 1 - \frac{ad_s}{bq}\eta^s + \text{terms of higher order.}$$

If then we put

$$[9] \qquad t = \eta[Q_1(\eta)]^{\frac{1}{q}} = \eta - \frac{ad_s}{bq}\eta^{s+1} + \cdots,$$

by reverting this series we have

$$[10] \qquad \eta = t + \frac{ad_s}{bq}t^{s+1} + \cdots = P(t), \text{ say,}$$

and from this it is seen that a positive quantity K may be so determined that for all values of t such that $|t| < K$ the above power-series in t converges. This power-series when written for η in the equation [9] identically satisfies it.

If, further, we raise the equation [9] to the qth power and multiply it by $-\frac{b}{a}$, we have

$$-\frac{b}{a}t^q = -\frac{b}{a}\eta^q Q_1(\eta) = -\frac{b}{a}\eta^q + d_s\eta^{q+s} + \cdots = Q(\eta) \text{ above;}$$

and this equation will be an identical one if for η we write the power-series $P(t)$. The same is true of equation [8]; that is, we have the identity

$$G\left(-\frac{b}{a}t^q,\ P(t)\right) = 0$$

for all values t for which the series $P(t)$ is convergent.

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If we denote the radius of convergence of the series $P(t)$ by R and put $\xi = -\frac{b}{a}t^q$, where t is any one of the q-values of the qth root of $-\frac{a}{b}\xi$, we have the following theorem:

If $|\xi| < \left|\frac{b}{a}\right| R^q$, *so that the series*

$$[11] \qquad j\sqrt{-\frac{a\xi}{b}} + \frac{ad_s}{bq}\left(j\sqrt{-\frac{a\xi}{b}}\right)^{s+1} + \cdots$$

exist, j denoting any of the qth roots of unity, then this expression written for η causes the function $G(\xi, \eta)$ to vanish identically.

Furthermore, there is only one such convergent series in integral or fractional positive powers of ξ, without constant term, which when substituted for η in equation [1] causes that equation to vanish identically.

For if there were another such series in integral positive powers of $\xi^{\frac{1}{\mu}}$, say,

$$[12] \qquad \eta = c_\lambda \xi^{\frac{\lambda}{\mu}} + \cdots \quad (\lambda \geqq 1,\ c_\lambda \neq 0),$$

then in the manner given above we could express $\xi^{\frac{1}{\mu}}$, and consequently also ξ, through a power series in $\eta^{\frac{1}{\lambda}}$ which identically satisfied [1]; but besides the series [7] there exists no such series, and consequently there is no such series as [12] which is different from [11].

III. METHOD OF FINDING ALL SERIES FOR y WHICH BELONG TO A k-PLY SINGULAR POSITION*

142. In equation [1] let $dF(x_0, y_0), d^2F(x_0, y_0), \cdots, d^{k-1}F(x_0, y_0)$ be zero, so that this equation becomes

$$[13] \quad G(\xi, \eta) = \frac{1}{k!}d^kF(x_0, y_0) + \frac{1}{(k+1)!}d^{k+1}F(x_0, y_0) + \cdots + \frac{1}{N!}d^NF(x_0, y_0) = 0,$$

where N is the dimension of $F(x, y)$ with respect to x and y.

* Besides Stolz, p. 182, see also Puiseux, *Journ. de Math.*, 1st Series, Vol. XV, p. 365; Picard, *Traité* etc., Vol. I, p. 392; Hermite's preface to Appell et Goursat, *Fonctions Algébriques* etc.; Königsberger, *Elliptische Functionen*, Vol. I, p. 187 et seq.; etc.

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There is, consequently, a k-ple singularity at x_0, y_0, and we shall next show that we may derive all those convergent series without constant term which proceed in integral or fractional powers of ξ and which when substituted for η in [13] identically satisfy it, if we can derive corresponding series for any simple, double, up to $(k-1)$-ple position of any algebraic structure. In other words, the problem of deriving these series for a k-ple singularity is made to depend upon the derivation of such series for a position that is less than k-ple.

If for η in the homogeneous function* of the nth dimension

$$d^n F(x_0, y_0) = \Phi_n(\xi, \eta)$$

(it being supposed not identically zero) we write the series [12] and arrange in ascending powers of ξ, then if $\lambda = \mu$, this expression begins at least with ξ^n, and exactly with this term if $\Phi_n(1, c_\lambda)$ does not vanish. If $\lambda \neq \mu$, this expression begins with ξ^n only when this term in reality appears in $\Phi_n(\xi, \eta)$; otherwise with a term of higher or lower order than ξ^n according as $\lambda > \mu$ or $\lambda < \mu$.

If in [13] we next decompose the lowest differential

$$d^k F(x_0, y_0) = \Phi_k(\xi, \eta)$$

into its real or complex linear factors, we have

$$[14] \qquad \Phi_k(\xi, \eta) = \prod_{r=1}^{r=l} (\alpha_r \eta - \beta_r \xi)^{k_r},$$

where $k_1 + k_2 + \cdots + k_l = k$ and where one of the two coefficients α_r, β_r may be zero.

Assuming first that $\lambda = \mu$, if in the above expression we write

$$\eta = c_\lambda \xi + \cdots,$$

we see at once that for at least one value of r we must have

$$\alpha_r c_\lambda - \beta_r = 0.$$

* The method given by Weierstrass, *Werke*, Vol. IV, pp. 19 et seq., is essentially the same as that found here; see also Stolz, loc. cit.

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For if this were not the case, then $G(\xi, c_\lambda \xi + \cdots)$ would begin with ξ^k instead of vanishing identically.

If, next, $\lambda > \mu$, one of the quantities $\beta_1, \beta_2, \cdots, \beta_l$ must be zero; and if $\lambda < \mu$, then one of the quantities $\alpha_1, \alpha_2, \cdots, \alpha_l$ must vanish. For if they were all different from zero, then $G(\xi, c_\lambda \xi^{\frac{\lambda}{\mu}} + \cdots)$ begins with $\xi^{\frac{k\lambda}{\mu}}$.

If a series of the form [12], where $\lambda = \mu$, satisfies the equation [13], we shall have, if in [12] we write $\eta = (c_\lambda + \eta_1)\xi$, a relation between η_1 and ξ, viz., $\eta_1 = c_{\lambda+1}\xi^{\frac{1}{\lambda}} + c_{\lambda+2}\xi^{\frac{2}{\lambda}} + \cdots$.

The expression $G(\xi, (c_\lambda + \eta_1)\xi)$ contains the factor ξ^k, which may be neglected, so that η_1 satisfies the equation

$$G_1(\xi, \eta_1) = \frac{1}{\xi^k} G\big(\xi, (c_\lambda + \eta_1)\xi\big) = 0.$$

If in the series [12] (when $\lambda > \mu$) we write $\eta = \eta_1 \xi$, we find that the equation

$$G_1(\xi, \eta_1) = \frac{1}{\xi^k}(\xi, \eta_1 \xi) = 0$$

is satisfied by the series

$$\eta_1 = c_\lambda \xi^{\frac{\lambda-\mu}{\mu}} + \cdots.$$

If a series for η where $\lambda < \mu$ satisfies [13], we revert the process and make the substitution $\xi = \eta \xi_1$.

143. In giving the practical method of determining the series for η which satisfies [13] we must make a distinction between two cases: The function $\Phi_k(\xi, \eta)$ *either* may contain different linear factors to their respective powers *or* it is the kth power of one single such factor.

First case. Among the quantities $\alpha_1, \alpha_2, \cdots, \alpha_l$ there must be at least one which is not zero. For each α_i which does not vanish we put $\frac{\beta_i}{\alpha_i} = c^{(i)}$ and make in [13] the substitution

[15] $$\eta = (c^{(i)} + \eta_1)\xi.$$

We may then write

$$G\big(\xi, (c^{(i)} + \eta_1)\xi\big) = \xi^k G_1(\xi, \eta_1),$$

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where G_1 is an integral function in ξ and η_1 which vanishes for $\xi = 0$ and $\eta_1 = 0$. If, for example, $i = 1$, we have

$$[16] \qquad G(\xi, \eta_1) = \frac{1}{k!}(\alpha_1\eta_1)^{k_1}\prod_{r=2}^{r=l}[(\alpha_r c^{(1)} - \beta_r) + \alpha_r\eta_1]^{k_r} + \frac{\xi}{(k+1)!}\Phi_{k+1}(1, c^{(1)} + \eta_1) + \cdots + \frac{\xi^{N-k}}{N!}\Phi_N(1, c^{(1)} + \eta_1).$$

From this it is evident that the position $\xi = 0$, $\eta_1 = 0$ in the structure $G(\xi, \eta_1)$ is at most a k_1-ple singularity and consequently less than a k-ple, so that the problem may be regarded as solved, since, by hypothesis, when $k_1 < k$ we have supposed that we may derive all power-series which satisfy $G(\xi, \eta_1) = 0$. For η, through the formula $\eta = (c^{(1)} + \eta_1)\xi$, we have series arranged in integral or fractional positive powers of ξ which substituted in $G(\xi, \eta)$ cause this expression to vanish identically. Besides these series there are no other such series for η which begin with the term $c^{(1)}\xi$.

If in [15] we let r take all the values where $\alpha_r \neq 0$, we have in this way all those series for η, where $\lambda \equiv \mu$, which satisfy the equation $G(\xi, \eta) = 0$. Among the quantities $\alpha_1, \alpha_2, \cdots, \alpha_l$ there may be one, for example α_1, which is zero. If we consider η and ξ interchanged and then make in [13] the substitution $\xi = \eta\xi_1$, we may derive all series which proceed according to integral or fractional positive powers of η with constant term zero and which when written for ξ in the equation $G(\xi, \eta) = 0$ identically satisfy it, and whose initial term is $d_\mu\eta^{\frac{\mu}{\lambda}}$, where $\mu > \lambda$.

By reverting each of these series we may express η as series in terms of ξ which satisfy [13], where $\lambda < \mu$.

Further, we have all such series. For if [13] was solved by writing for η a series [12], then we also satisfy [13] by writing for ξ a series in integral positive powers of $\eta^{\frac{1}{\lambda}}$ whose initial term contains $\eta^{\frac{\mu}{\lambda}}$, where $\frac{\mu}{\lambda}$ is an improper fraction.

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Second case. Let $\Phi_k(\xi, \eta) = (\alpha\eta - \beta\xi)^k$ and suppose *first* that $\alpha \neq 0$. We make in [13] the substitution

$$\eta = \left(\frac{\beta}{\alpha} + \eta_1\right)\xi \qquad (i)$$

and have, after division by ξ^k, the new equation

$$[17] \quad G_1(\xi, \eta_1) = \frac{(\alpha\eta_1)^k}{k!} + \frac{\xi}{(k+1)!}\Phi_{k+1}\left(1, \frac{\beta}{\alpha} + \eta_1\right) + \cdots = 0.$$

If for this equation the position $\xi = 0$, $\eta_1 = 0$ is less than a k-ple singularity the problem is by hypothesis solved, or if it remains a k-ple singularity and if the polynomial of the terms of the kth order in ξ and η_1 may be decomposed into *different* linear factors, we may proceed as in the first case. It may happen, however, that the position $\xi = 0$, $\eta_1 = 0$ is a k-ple singularity whose terms again form the kth power of a linear expression in ξ and η_1 which must necessarily be $\frac{1}{k!}(\alpha\eta_1 - \beta_1\xi)^k$.

If, further, we write in [17] η_2 instead of η_1, where η_2 is defined by the equation

$$\eta_1 = \left(\frac{\beta_1}{\alpha} + \eta_2\right)\xi, \qquad (ii)$$

the expression will be divisible by ξ^k, so that we may write

$$G_1\left(\xi, \left(\frac{\beta}{\alpha} + \eta_2\right)\xi\right) = \xi^k G_2(\xi, \eta_2),$$

where

$$G_2(\xi, \eta_2) = \frac{(\alpha\eta_2)^k}{k!} + \xi H_2(\xi, \eta_2),$$

$H_2(\xi, \eta_2)$ being an integral function of ξ and η_2.

Noting (i) and (ii) it is seen that if there is for η a series of the form

$$[18] \qquad \eta = \frac{\beta}{\alpha}\xi + \frac{\beta_1}{\alpha}\xi^2 + c_\lambda \xi^{2+\frac{\lambda}{\mu}} + \cdots,$$

then for $\xi = 0$ the quantity η_2 introduced above must be zero, and η_2 must belong to those series that vanish with ξ and which are obtained from the equation $G_2(\xi, \eta_2) = 0$.

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This equation may be solved as above for η_2 if the position $\xi = 0$, $\eta_2 = 0$ for the structure $G_2(\xi, \eta_2) = 0$ is less than a k-ple singularity or if it is a k-ple singularity in which the terms of the kth order do not constitute the kth power of a linear function of ξ and η_2. We further have all series, proceeding according to powers of ξ without constant term, which when substituted in [13] satisfy it, if we solve the equation

$$G(\xi, \eta_2) = 0$$

with respect to η_2 in all possible ways through power-series in ξ without constant term and substitute these series for η_2 in the expression (cf. (*i*) and (*ii*))

$$\eta = \frac{\beta}{\alpha}\xi + \frac{\beta_1}{\alpha}\xi^2 + \eta_2\xi^2.$$

But if the position $\xi = 0$, $\eta_2 = 0$ is a k-ple singularity in the structure $G_2(\xi, \eta_2) = 0$, and if the terms of the kth order form the kth power of a linear expression in ξ, η_2, which must have the form $\frac{1}{k!}(\alpha\eta_2 - \beta_2\xi)^k$, we must write η_3 instead of η_2, where η_3 is defined by

$$\eta_2 = \left(\frac{\beta_2}{\alpha} + \eta_3\right)\xi, \qquad (iii)$$

and proceed in a similar manner as above.

Continuing in this manner it is evident that if $\alpha \neq 0$ we may derive all power-series in ξ without constant term which written for η in the equation [13] identically satisfy it, if through a series of transformations

$$[19]\ \eta = \left(\frac{\beta}{\alpha} + \eta_1\right)\xi,\ \eta_1 = \left(\frac{\beta_1}{\alpha} + \eta_2\right)\xi, \cdots,\ \eta_{h-1} = \left(\frac{\beta_{h-1}}{\alpha} + \eta_h\right)\xi$$

we may from the given equation $G(\xi, \eta) = 0$ derive an equation $G_h(\xi, \eta_h) = 0$ whose left-hand side does *not* begin with the kth power of a linear expression in ξ and η_h.

We must finally come to such an equation if $F(x, y)$ and $\frac{\partial F}{\partial y}$ have no divisor in common. For, since the factor ξ^k appears with each of the substitutions [19], it is easily shown that the integer

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h in [19] cannot pass a fixed limit. For if F is of the nth degree in y, we may always find two integral functions U and V in x and y where U is at most of the $(n-1)$st degree in y and V at most of the $(n-2)$d degree in y such that there exists the identical relation

$$[20] \qquad VF(x, y) + U\frac{\partial F}{\partial y} = D(x),$$

where $D(x)$ is an integral function in x.

Furthermore, since

$$G\left(\xi, \left(\frac{\beta}{\alpha} + \eta_1\right)\xi\right) = \xi^k G_1(\xi, \eta),$$

$$G_1\left(\xi, \left(\frac{\beta_1}{\alpha} + \eta_2\right)\xi\right) = \xi^k G_2(\xi, \eta),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$G_{h-1}\left(\xi, \left(\frac{\beta_{h-1}}{\alpha} + \eta_h\right)\xi\right) = \xi^k G_h(\xi, \eta_h),$$

it is seen that

$$F(x_0 + \xi, y_0 + \eta) = G(\xi, \eta) = \xi^{hk} G_h(\xi, \eta_h).$$

We also note from the formula

$$F(x, y + v) = F(x, y) + \frac{\partial F}{\partial y} v + \cdots,$$

if we make the substitution $x = x_0 + \xi$, $y = y_0 + \eta$, since

$$F(x_0 + \xi, y_0 + \eta) = G(\xi, \eta),$$

that
$$G(\xi, \eta + v) = G(\xi, \eta) + \left[\frac{\partial F(x, y)}{\partial y}\right]_{\substack{x = x_0 + \xi \\ y = y_0 + \eta}} \cdot v + \cdots.$$

Expanding the left-hand side of this expression, it is seen that

$$\left[\frac{\partial F(x, y)}{\partial y}\right]_{\substack{x = x_0 + \xi \\ y = y_0 + \eta}} = \frac{\partial G(\xi, \eta)}{\partial \eta} = \xi^{hk} \frac{\partial G_h}{\partial \eta_h} \frac{\partial \eta_h}{\partial \eta} = \xi^{h(k-1)} \frac{\partial G_h}{\partial \eta_h}.$$

It follows that after the substitution of

$$x = x_0 + \xi, \; y = y_0 + \eta, \quad \text{where from [19]}$$

$$[21] \qquad \eta = \frac{\beta}{\alpha}\xi + \frac{\beta_1}{\alpha}\xi^2 + \cdots + \left(\frac{\beta_{h-1}}{\alpha} + \eta_h\right)\xi^h,$$

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INDEX

(The figures refer to the pages)

Abelian transcendents, 169
Algebra, fundamental theorem of, 165
Algebraic curve expressed through power series, 36, 53
Algebraic function, its development in series, 177; at a singular point, 180 et seq.
Algebraic structures, 176 et seq., 181, 187
Ambiguous case, the, iv, 27. *See* Semi-definite form
Analytic dependence, 166 et seq., 169
Analytic function, 73; defined, 138
Analytic structure, 74, 174
Appell, 180
Area, maximum area, 143
Asymptotic approach, 138
Auxiliary variable, 101, 173

Baltzer, 2
Bauer, 107
Bertrand, v, 33
Biermann, 136, 155, 168
Bôcher, 148
Bohlmann and Schepp, iv
Bois-Reymond, Paul du, 2, 7, 74
Bolzano, 12, 136
Borchardt, 107
Boundary, 136 et seq., 156 et seq.
Brand, 126 et seq.
Burnside, 89, 106

Calculus of variations, iv, 171
Cantor, *Geschichte* etc., 15
Cartesian Oval, 133
Cauchy, 3, 7, 92
Cavalieri, 27
Center of curvature, 117, 125
Christoffel, 107
Complete differential quotient, 6
Contact of indefinitely high order, 36
Continuation of an analytic function, 174
Continuous function, 161, 162
Convergence, 166, 167, 168, 172, 178
Cremona, 144
Curvature, 117
Cusps, appearance of, 30
Cylinder, trace of, 31

Dantscher, Victor von, v, 36, 69; method of, 39, 62 et seq., 72
Definite form, 19; necessary condition, 21, 49, 50, 51, 64, 68, 82, 83, 91, 92, 109, 111, 114; conditions for, 91, 103
Derivative, existence of a, 161, 162, 163, 166
Descartes, iv, 165
Determinant, the sign of the, 25 et seq., 28, 29, 30, 32, 33, 51, 52, 59, 60, 83, 85 et seq., 90, 91, 92, 93, 97, 100, 107, 111; orthogonal, 149
Differentiation, one-sided, iv, 7, 11 et seq.
Dini, 12, 136, 167
Distinctness as characteristic of an extreme, 37, 38, 47, 50
Double curve, 54
Double point, 101; with distinct tangents, 29; isolated, 29, 30, 31

Element of a complete structure, 174
Equation of secular variations, 107
Euclid, iii, 15, 135
Euler, 16, 18, 107; theorem of, for homogeneous functions, 84, 155
Exceptional cases involving a squared factor, 54, 58, 68, 97
Existence of an extreme, proof of, 135, 146
Extraordinary cases of extremes, iv, 19
Extraordinary maxima or minima, 1, 6 et seq., 17, 19, 43 et seq., 74
Extreme, or extreme value, v, 2, 53, 54; criteria for, 4, 6, 26, 92. *See* Maxima and minima
Extreme curves, 53, 54

Failure of general criterion, 55
Fallacious conclusions. *See* Incorrectness of earlier theories
Fermat, iii, iv, 15, 132; method of determining maximum and minimum, iii
Form. *See* Definite form
Fourier, iii
Fourier series, 74

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Fractional powers, 178, 182
Fuchsian functions, 171
Function, rational, 166; one-valued, 166; many-valued, 169
Function-element, 138, 174
Fundamental theorem of algebra, 49

Gauss, 19, 86, 99; principle of, 151 et seq.
Genocchi-Peano, 1
Geometrical interpretations, 6, 24, 31, 46, 69, 71, 97, 125
Geometrical mechanics, 139
Geometry of numbers, 174
Gergonne, 19
Goursat, 6, 23, 27, 29, 31, 126, 170, 177, 180
Greatest value, 1, 48, 94. *See* Upper and lower limits

Hachette, 107
Hadamard, 147, 148, 160
Hancock, 123, 166
Hankel, 74
Harkness, 12
Hermite, 89, 106, 180
Hilbert, 170
Homogeneous functions, 49, 155
Homogeneous quadratic forms, 82, 85, 103 et seq.; expressed as a sum of squares, 86, 89, 91; with subsidiary conditions, 114
Hudde, 165
Huygens, 16
Hypergeometric series, 170

Improper maxima and minima. *See* Maxima and minima
Incorrectness of earlier theories, 33 et seq., 52
Indefinite form, 19, 49, 50, 51, 64, 68, 82, 106, 116
Indeterminate coefficients, 172
Inflection, point of, 6
Integral rational function, 168
Isolated point, 29, 31

Jacobi, 107
Jordan, 75

Königsberger, 180
Kronecker, 106
Kummer, 106, 107

Lagrange, iii, v, 4, 18, 22, 26, 33, 43, 77, 86, 92, 99, 107, 114, 127, 131, 148, 172
Laplace, iii, 107
Least squares, 26
Least value, 1, 50, 94. *See* Upper and lower limits
Left-hand differential quotient, 7, 11
Legendre, 135
Leibnitz, 3, 15
Limitation expressed through an equation, 150
Lipschitz, 2, 74
Lower limit, 63, 94, 104, 136. *See* Upper limit
Lüroth. *See* Dini

Maclaurin, iii, 3, 4, 15, 22, 77
Maxima and minima (*see also* Extreme value), one of the most admirable applications of fluxions, iv; conditions for, iv, 4, 40, 99; inaccuracies in, v; maximum defined, 1; minimum, 1; ordinary (*see under* Ordinary etc.); extraordinary (*see under* Extraordinary etc.); proper, 2, 5, 11, 17, 23, 26, 44, 45, 60, 61, 63, 74, 75; improper, 2, 5, 17, 23, 26, 31, 50, 59, 60, 63, 75, 140 et seq., 164; absolute, 2; relative, 2, 21, 96 et seq.; criteria for, 4, 7–12, 40–42, 43 et seq., 48, 51, 55, 64, 67, 68, 77, 80, 81, 82, 92, 100, 102, 115, 116; geometrical interpretation of, 6, 46, 71; erroneous criteria, 33; condition for proper extremes, 40, 42; condition for improper extremes, 41, 42, 140 et seq.; criteria for relative maxima and minima, 115
Mayer, iv, v, 2, 79
Mechanics, problems in, 139, 150; derivation of the ordinary equations of, 152
Minimal surfaces, 123
Minkowski, 174
Morley. *See* Harkness

Neighborhood of, in the, 65, 173
Newton, discoverer of the calculus, iii

One-sided differential quotient, 7, 11 et seq.
Orbits of planets, 107
Order of a curve, 54
Ordinary maxima and minima, 1 et seq., 17 et seq. *See* Maxima and minima
Osculating circle, 117
Osgood, 167, 170

Panton. *See* Burnside
Pappus, 15, 164
Pascal, *Exercici* etc., 11; *Repertorium* etc., 130

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Peano, iv, v, 2, 6, 12, 18, 21, 31, 33, 34, 52, 61, 68, 94
Pendulum, 150
Petzval, 107
Picard, 170, 180
Pierpont, 3, 7, 15, 34, 37, 177
Poincaré, 170, 180
Poison, 107
Polygon. *See* Regular polygon
Position, 135 et seq.
Power-series, 171 et seq., 175, 179
Proper maxima or minima. *See* Maxima and minima
Puiseux, 180

Quadratic form, 19; expressed as a sum of squares, 86 et seq., 89; application of, 92 et seq. *See* Homogeneous quadratic forms

Radius of curvature, 117
Realm, 135, 174
Reflection of a ray of light, 126 et seq.
Refraction of a ray of light, 131 et seq.
Regiomontanus, 16
Region of convergence, 167, 168
Regular function, 73
Regular point, 177
Regular polygon, 140, 142 et seq., 147
Relative maxima and minima. *See* Maxima and minima
Reversion of series, 153 et seq.
Richelot, 106
Right-hand differential quotient, 7, 11
Roots of unity, 180

Salmon, 107, 120, 122
Scheeffer, v, 19, 27, 35, 36, 39, 46, 48, 50, 62, 70
Scheeffer's method, 37
Scheeffer's theorem, 43, 46, 55, 59, 60, 61, 62, 70, 72
Scheeffer's theory, 43 et seq.
Schepp. *See* Bohlmann; *see also* Dini
Secular variations, equation of, 107
Semi-axes of a central section, 165
Semi-definite case, iv
Semi-definite form, 19, 49, 50, 51, 52, 64, 65, 68, 70 et seq., 82, 83, 92, 93, 106, 116
Serret, v, 33, 104, 106, 136
Severus, 16
Shortest distance to a given surface, 101, 123
Simple point, 177
Simpson, 16
Singular point, 164, 177, 180 et seq., 183, 184
Sluse, René F. W. de, 16
Smallest value, 1
Smith, Edward, 107
Spherical triangle, 135
Squared factor. *See* Exceptional cases
Stolz, v, 2, 6, 11, 14, 43, 45, 46, 50, 60, 70, 79, 91, 100, 136, 155, 164, 180, 181, 187, 189
Stolzian theorems, 39 et seq., 55, 58, 70, 72
Stolz's added theorem, 45, 60
Structure of the first kind etc., 175
Sturm's theorem, 51, 106
Surfaces of second degree, 107
Sylvester, 89, 107
System of m equations, solution of, 171 et seq.

Tangent, parallel to x-axis, 6; common to two curves, 34, 35, 36
Tangential plane, 28, 31
Tartaglia, 16
Taylor's development in series, v, 4, 5, 9, 10, 19, 24, 33, 75, 79, 80, 97, 152, 159
Taylor-Lagrange theorem, 43, 47, 77
Todhunter, 33
Transcendental curves, 36
Transcendental functions, 167, 169

Uniform. *See* Convergence
Upper and lower limits, 2, 12 et seq., 55, 57, 94, 104, 136, 137

Variations, calculus of, iv, 171
Von Dantscher. *See* Dantscher
Voss, 2

Weierstrass, iv, 73, 79, 86, 107, 138, 167, 168, 169, 174, 181
Wilson, E. B., 12
Wirtinger, 148

Zajaczkowski, 106
Zenodorus, 142, 147

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